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Dynamics of the Open BCS Model

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The dynamics of the strong coupling BCS model, considered as an open system interacting with a thermal bath, is solved rigorously and explicitly in the weak coupling limit and in the infinite-volume limit. The BCS system goes from the normal phase to the ordered phase by bifurcation. Fluctuations around trajectories of intensive observables are Gaussian and Markovian. Thermodynamic phases are global attractors in the physical domain. Structural stability is discussed. The model provides an example of a nonequilibrium statistical mechanical system with phase transition whose irreversible macroscopic dynamics can be calculated exactly from the underlying Hamiltonian quantum mechanics.

KEY WORDS: Strong coupling BCS model; macroscopic states; heat bath; dissipative semigroup; bifurcation; Liapunov function; stability; nonequilibrium thermodynamics; fluctuations.

1. INTRODUCTION

Some progress has been made in recent years in establishing a precise link between microscopic dynamical laws and the thermodynamics of irreversible processes. In appropriate limiting situations, master and transport equations can be obtained in a mathematically controlled way from the underlying Hamiltonian mechanics. We have, for instance, a derivation of the master equation[(1,2)] and of a quantum transport equation[(3,4)] in the weak coupling limit (van Hove limit), a treatment of the laser in the singular coupling limit,[(5,6)] and, for classical systems, the Boltzmann equation in the Grad limit[(7)] and the Vlasov equation in the mean field limit.[(8)] In this work, we study another example of a dissipative system, the open BCS model, whose dynamics originates rigorously from Hamiltonian quantum mechanics in the weak coupling limit. (For a simpler system, the open Ising model, see Ref. 9 and further developments in Ref. 10).

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How to formulate a precise microscopic theory without the recourse to such limiting procedures is certainly a fundamental problem, which involves not only technical, but also conceptual difficulties. However, it is worth to explore the weak coupling limit situation thoroughly, for the following reason: it provides a microscopic justification for many of the features of the thermodynamics of irreversible processes (for a review see Ref. 11). Moreover, since there does not exist a general prescription (analogous to the Gibbs prescription in the theory of equilibrium) for computing the non-equilibrium behavior of macroscopic systems, we have to rely on the Liouville–von Neumann equation of motion as a last resort. It is therefore of interest to investigate as fully as possible models for which a link without mathematical gap can be established with the microscopic mechanics.

We consider here the strong coupling version of the BCS model as an open system coupled with an external agency. This external agency is responsible for two basic mechanisms: the creation or destruction of electron pairs and the scattering of pairs. The first mechanism, which is not gauge invariant, causes a variation of the number of pairs. As a result, the number of pairs and the order parameter obey a coupled set of equations of motion. The scattering of pairs, which is a gauge-invariant interaction, affects only the evolution of the order parameter. We have chosen to treat here the first case as being most illustrative and leading to the richest dynamical structure. Of course, many variations of the model can be worked out with analogous results.

In Section 2, we give the free dynamics of the BCS model in a closed form, which becomes exact in the infinite-volume limit. Although the dynamics of the strong coupling BCS model has been extensively studied,[12–14] to our knowledge this explicit form has not appeared in the literature. A solution of this dynamical problem valid in a large class of nonequilibrium states is a necessary preliminary to the study of the open system. We specify in Section 3 the external system and its interaction with the BCS model: it is a quasi-free thermal bath essentially characterized by the KMS relation. After a brief review of the framework of the weak coupling limit, we establish the equations of motion of the relevant intensive observables in the infinite-volume limit. The number of pairs and the gap parameter obey a two-dimensional autonomous differential system. It is interesting to note that the simple quadratic nonlinearity of the free BCS Hamiltonian produces a highly nonlinear set of equations of motion for the open system.

Section 4 is devoted to the study of the corresponding flow of intensive observables. A global analysis is made possible by the fact that the free energy is a Liapunov function of our differential system. It is shown that above the critical temperature the normal phase is a global attractor in the physical domain. At the critical temperature, there is a bifurcation, the normal phase becoming unstable. Below the critical temperature the superconducting phase becomes an attractor for all initial conditions that do not belong to the (unstable) manifold where the gap parameter vanishes. We show that the concept of structural stability is relevant in this context[2]: Our dynamical system is structurally stable if its only equilibria are the thermodynamic ones. In this sense, the qualitative aspects of trajectories are independent of the detailed features of the coupling functions with the bath. Structural stability is lost as soon as there exists accidental equilibria due to a peculiarity of the coupling with the bath. Finally, we examine how the symmetry breaking of gauge invariance occurs dynamically, by the study of the complex order parameter. In the case of a non-gauge-invariant interaction with the bath, the two-dimensional order parameter undergoes a Hopf bifurcation, and it moves asymptotically on an attracting orbit constituted by the set of extremal equilibrium states.

The dynamical description is completed in Section 5 by the study of the behavior of fluctuation observables. The main result is that the generator of the evolution of the probability distribution for fluctuation observables converges in the infinite-volume limit to a Fokker–Planck-type generator. The associated Fokker–Planck equation is exactly that which corresponds to the process obtained by assuming that fluctuation observables follow the linearized equation of motion around classical trajectories with a Markovian Gaussian (but not stationary) random force. Regression of fluctuations and dynamical instabilities can easily be discussed, owing to our previous knowledge of the flow of intensive observables.

Since the model is solvable and provides the dynamics of the phase transition, it is of interest to compare our results with the existing phenomenological dynamical theories of critical phenomena (for a recent review see Ref. 15). Although our differential system is not of gradient form, we show in Section 6 that in the neighborhood of the critical temperature and for small values of the order parameter it reduces to the usual Ginzburg–Landau time-dependent theory. Of course no mode–mode coupling occurs here, the strong coupling BCS model being strictly mean field. We have given the main steps of all calculations, but skipped parts of them that are straightforward and lengthy. Some more technical points are relegated to appendices.

The model is treated in a spirit very similar to that of Hepp and Lieb in their work on the laser.[5,6] Every step on the way from microscopic to macroscopic dynamics can be stated and discussed without recourse to any ad hoc statistical assumption. One difference is that the laser makes its phase transition out of equilibrium when the coupling parameters to the

[2] See Ref. 10 for a proof of structural stability in a generalized open Ising–Weiss model.

reservoirs are varied. Here the bifurcation occurs when the temperature of the initial state of the bath is lowered. Taking advantage of the simplicity of the singular coupling limit, they work in the framework of Heisenberg equations of motion, and are thus led to a Langevin description of random forces. We preferably use the theory of master equations, where the KMS relation can be more easily exploited in the weak coupling limit, and we end with a Fokker–Planck description of fluctuations. One should note in both cases the involved limits and the order in which they are taken. First of all the thermodynamic limit is taken on the external system, keeping the BCS model finite. Then one describes its evolution in terms of the corresponding quantum dynamical semigroup obtained by the weak coupling limit. Finally the infinite-volume limit of the BCS system is also taken for a suitable class of states. We emphasize the importance of this last step in relation to the mathematically well-developed theory of quantum dynamical semigroups (see Refs. 16 and 17 for references). It is only in the infinite-volume limit that the dynamical content of such a semigroup is clearly and plainly revealed. Here the relevant dynamical information contained in the semigroup appears in the structure of the associated flow discussed in Section 4 (equilibria, basins of attraction, bifurcations). These characteristic features remain completely hidden if one looks only at the form that the generator of the semigroup takes for a finite volume. The situation is very much the same as in equilibrium theory, where the possibility of having various phases and critical phenomena occurs only after the thermodynamic limit.

To conclude this introduction, we can say that the open BCS system goes from disorder (normal phase) to order (superconducting phase) by bifurcation, the normal phase becoming an unstable thermodynamic branch below the critical temperature. As the laser, it provides a simple physical example, not only phenomenological, but rooted in the microscopics, of these self-organizing systems, which are so convincingly advocated in the recent books by Nicolis and Prigogine[18] and Haken.[19]

2. THE FREE EVOLUTION OF THE STRONG COUPLING BCS MODEL

2.1. Quasi-Spin Formulation

In this work we shall use the quasi-spin formulation of the BCS model, which we briefly review.[20]

The Hamiltonian is

$$H = \epsilon \sum_{p\in\Omega} (\sigma_p{}^0 + I) - (\mu/V) \sum_{p,q\in\Omega} \sigma_p{}^+\sigma_q{}^- \tag{1}$$

where σ_p^α ($\alpha = +, -, 0$) are the Pauli matrices

$$\sigma_p^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_p^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_p^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_p^+, \sigma_q^-] = \delta_{pq}\sigma_p^0, \qquad [\sigma_p^0, \sigma_q^+] = 2\delta_{pq}\sigma_p^+ \tag{2}$$

σ_p^+ and σ_p^- are the creation and annihilation operators for Cooper pairs of electrons with momentum p. The electronic system is quantized in a box of volume $V = L^3$ with periodic boundary conditions, and the momenta take the values

$$p = \{2\pi n_i/L, \quad n_i = \text{integer}\}$$

The summation in (1) is restricted to a finite region Ω of momentum space in the neighborhood of the Fermi surface, so that sums are finite in (1) and H acts in the finite tensor product $\prod_{p\in\Omega} \mathbb{C}^2$ of single spin spaces.

Finally, the kinetic energy ϵ of the electrons has been taken independent of p in the region Ω.

Denoting by $N = \sum_{p\in\Omega} 1$ the number of pair levels at volume V, we have N proportional to V, and after a redefinition of the coupling constant μ, we may write the Hamiltonian in the form

$$H_N = \epsilon \sum_p \sigma_p^0 - (\mu/N) \sum_{pq} \sigma_p^+ \sigma_q^-, \qquad \mu > 0 \tag{3}$$

In (3) we have dropped the constant ϵN term, which is irrelevant for the dynamics, and summations are always understood to run in Ω.

The interesting physical quantities are the intensive observables

$$S_N^\alpha \equiv (1/N) \sum_p \sigma_p^\alpha, \qquad \alpha = +, -, 0 \tag{4}$$

where $\frac{1}{2}(S_N^0 + 1)$ is the density of Cooper pairs and S_N^+ is the (complex) order parameter.

The intensive observables are bounded operators

$$\|S_N^\alpha\| \leqslant (1/N) \sum_p \|\sigma_p^\alpha\| \leqslant 1 \tag{5}$$

and commutators $[S_N^\alpha, S_N^\beta]$ and $[S_N^\alpha, \sigma_p^\beta]$ are always of order $O(1/N)$ in operator norm.

More generally, we shall consider the set of intensive "one-pair" observables $S_N^\alpha(g)$ defined as follows:

$$S_N^\alpha(g) = (1/N) \sum_p g(p) \sigma_p^\alpha \tag{6}$$

where $g(p)$ belongs to the class $C^0(\Omega)$ of continuous functions on Ω. Then

$\|S_N{}^\alpha(g)\| \leqslant \sup_p|g(p)|$ and commutators are also $O(1/N)$. In particular, for $g(p) = 1$ on Ω, $S_N{}^\alpha(1) = S_N{}^\alpha$.

One can also view $H_N = N(\epsilon S_N{}^0 - \mu S_N{}^+ S_N{}^-)$ as the Hamiltonian of a mean field XY model in an external magnetic field in the z direction. This interpretation is useful for geometric insight.

2.2. The Free Dynamics

We shall base our analysis of the dynamics on the Heisenberg equations of motion:

$$\begin{aligned}
\frac{d}{dt}\sigma_{pN}{}^+(t) &= 2i\epsilon\sigma_{pN}{}^+(t) + i\mu S_N{}^+(t)\sigma_{pN}{}^0(t) \\
\frac{d}{dt}\sigma_{pN}{}^-(t) &= -2i\epsilon\sigma_{pN}{}^-(t) - i\mu\sigma^\circ{}_{pN}(t)S_N{}^-(t) \qquad (7) \\
\frac{d}{dt}\sigma_{pN}{}^0(t) &= 2i\mu\sigma_{pN}{}^+(t)S_N{}^-(t) - 2i\mu S_N{}^+(t)\sigma_{pN}{}^-(t)
\end{aligned}$$

where the volume dependence of

$$\begin{aligned}
\sigma_{pN}{}^\alpha(t) &= \exp(iH_N t)\,\sigma_p{}^\alpha\exp(-iH_N t) \\
S_N{}^\alpha(t) &= \exp(iH_N t)\,S_N{}^\alpha\exp(-iH_N t)
\end{aligned} \qquad (8)$$

is denoted by the index N.

Since the evolution is unitary, equal time commutation relations are preserved. Commutators between intensive observables are still $O(1/N)$ and

$$\|S_N{}^\alpha(t)\| \leqslant 1 \qquad (9)$$

We remark that the nonlinearity of the right-hand side of (7) enters only through the intensive observables $S_N{}^\alpha(t)$, and we write (7) in a more compact form, introducing the three-component operator $\sigma_{pN}(t) = \{\sigma_{pN}^\alpha(t), \alpha = +, -, 0\}$:

$$\frac{d}{dt}\sigma_{pN}(t) = i\Gamma_N(t)\sigma_{pN}(t) \qquad (10)$$

where $\Gamma_N(t) = \{\Gamma_N^{\alpha\beta}(t)\}$ is a 3×3 matrix acting on $\sigma_{pN}(t)$. Its entries, which are defined in such a way as to reproduce (7), depend only on the intensive observables $S_N{}^\alpha(t)$. [One can always define $\Gamma_N(t)$ to act on the left, using equal time commutation rules, for instance

$$\frac{d}{dt}\sigma_{pN}^-(t) = -2i\left(\epsilon - \frac{\mu}{N}\right)\sigma_{pN}^-(t) - i\mu S_N{}^-(t)\sigma_{pN}^0(t)$$

Hence $\Gamma_N^{--}(t) = -2(\epsilon - \mu/N)$, $\Gamma_N^{-0}(t) = -\mu S_N{}^-(t)$, and so on.]

The main observation is that the differential equations for the intensive observables that we deduce from (7)

$$\frac{d}{dt} S_N{}^+(t) = 2i\epsilon S_N{}^+(t) + i\mu S_N{}^+(t) S_N{}^0(t), \qquad \frac{d}{dt} S_N{}^0(t) = 0 \tag{11}$$

can be solved explicitly with

$$S_N{}^0(t) = S_N{}^0, \qquad S_N{}^+(t) = S_N{}^+ \exp[(2i\epsilon + i\mu S_N{}^0)t] \tag{12}$$

The constancy in time of $S_N{}^0$ expresses the conservation of the pair number, or the invariance under rotations around the z axis.

Therefore, $\Gamma_N(t)$ is a known function of time and time-zero intensive observables $S_N{}^\alpha$. Then the solution of (7) is given by the iteration series

$$\begin{aligned} \sigma_{pN}(t) &= \sum_{n=0}^{\infty} (i)^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_1 \cdots dt_n \, \Gamma_N(t_1) \cdots \Gamma_N(t_n) \sigma_p \\ &\equiv T\left(\exp i \int_0^t \Gamma_N(t')\, dt'\right)\sigma_p \end{aligned} \tag{13}$$

The series converges in the operator norm uniformly with respect to N since by (9) the matrix elements of $\Gamma_N(t)$ are operators bounded uniformly in N as well as in time.

2.3. Macroscopic States and Free Evolution in the Thermodynamic Limit

For a suitable class of states, the dynamics given by the series (13) simplifies considerably in the thermodynamic limit $N \to \infty$, and can be written in a closed form. States leading to a macroscopic description of the system are those for which average values of intensive observables are well defined and uncorrelated as $N \to \infty$. We incorporate this property in a definition.

Let $S(g) = (1/|\Omega|) \int_\Omega S(p) g(p)\, d^3p$ be three classical functionals, where $S(p) = \{S^\alpha(p), \alpha = +, -, 0\}$ are three given continuous functions on Ω satisfying $S^+(p) = (S^-(p))^*$ and $|S^\alpha(p)| \leqslant 1$, and consider a sequence of states ρ_N (ρ_N is a density matrix on $\mathbb{C}^{2 \otimes N}$). The sequence ρ_N is *macroscopic at* $S(g)$ if

$$\lim_{N\to\infty} \rho_N\Big[\prod_r S_N^{\alpha_r}(g_r)\Big] = \prod_r S^{\alpha_r}(g_r) \tag{14}$$

for all monomials $\prod_r S_N^{\alpha_r}(g_r)$ of intensive observables (6), where $\alpha_r = +, -, 0$, and $g_r \in C^0(\Omega)$.[3] (A class of macroscopic states is given in Appendix A.)

[3] We note $\rho_N(A_N) = \mathrm{Tr}\, \rho_N A_N = \langle A_N \rangle$.

The simplification that occurs in the thermodynamic limit is due as usual to the fact that intensive observables commute, allowing us to treat them as c-numbers.

For this reason, one is led to consider the semiclassical set of differential equations associated to (7):

$$\begin{aligned} \frac{d}{dt}\sigma_p^+(t) &= 2i\epsilon\sigma_p^+(t) + i\mu S^+(t)\sigma_p^0(t) \\ \frac{d}{dt}\sigma_p^-(t) &= -2i\epsilon\sigma_p^-(t) - i\mu S^-(t)\sigma_p^0(t) \\ \frac{d}{dt}\sigma_p^0(t) &= 2i\mu S^-(t)\sigma_p^+(t) - 2i\mu S^+(t)\sigma_p^-(t) \end{aligned} \tag{15}$$

where $S^+(t)$ and $S^-(t)$ are now given time-dependent classical functions

$$S^+(t) = S^+ \exp[(2i\epsilon + i\mu S^0)t], \qquad S^-(t) = (S^+(t))^* \tag{16}$$

and the $\sigma_p^\alpha(t)$ are the spin operators defined as solutions of (15) with initial values $\sigma_p^\alpha(t = 0) = \sigma_p^\alpha$. We notice that in (15) the spin operators for different values of p are no longer coupled, and are independent of N. Therefore (15) can easily be solved in closed form as a quantum mechanical problem in a single spin space (see Section 2.4).

On the other hand, we can write (15) in the same manner as (10):

$$\frac{d}{dt}\sigma_p(t) = i\Gamma(t)\sigma_p(t) \tag{17}$$

where $\sigma_p(t) = \{\sigma_p^\alpha(t); \alpha = +, -, 0\}$, and $\Gamma(t) = \{\Gamma^{\alpha\beta}(t)\}$ is the 3×3 matrix whose entries are the same as those of $\Gamma_N(t)$ but with the operators $S_N^\alpha(t)$ of (12) replaced by the classical functions $S^\alpha(t)$ of (16) and terms $O(1/N)$ dropped. Hence, solving (17) by iteration, we have

$$\sigma_p(t) = T\left(\exp i\int_0^t \Gamma(t')\,dt'\right)\sigma_p \tag{18}$$

The classical propagator $T(\exp i\int_0^t \Gamma(t')\,dt')$ is defined by the same absolutely convergent series as in (13) with $\Gamma_N(t)$ replaced by $\Gamma(t)$.

The main result concerning the free dynamics of the model defined by the Hamiltonian H_N is that it is given for macroscopic states and as $N \to \infty$ by the solutions of (15).

Equivalently, the full quantum mechanical propagator can always be replaced by its classical equivalent (18) as $N \to \infty$.

Proposition 1. Let ρ_N be a sequence of macroscopic states at $S(g)$ and let $\rho_N{}^t$ evolve with the Hamiltonian H_N; then the sequence $\rho_N{}^t$ is macroscopic at $S(g, t) = T(\exp i\int_0^t \Gamma(t')\,dt')S(g)$.

Proof. Let $F(S_N{}^\alpha) = \sum_{n=0}^{\infty} c_n(S_N{}^\alpha)^n$ be a function of $S_N{}^\alpha$ defined by a norm convergent power series; then $\lim_{N\to\infty} \rho_N(F(S_N{}^\alpha)) = F(S^\alpha)$. Indeed,

$$\rho_N(F(S_N{}^\alpha)) - F(S^\alpha) = \sum_{n=0}^{M} c_n[\rho_N(S_N{}^\alpha)^n - (S^\alpha)^n] + \rho_N\left[\sum_{n=M+1}^{\infty} c_n(S_N{}^\alpha)^n\right] - \sum_{n=M+1}^{\infty} c_n(S^\alpha)^n \quad (19)$$

Choosing M large enough, using $\mathrm{Tr}\, \rho_N = \rho_N(I) = 1$ and (5), we can make

$$\left|\rho_N\left[\sum_{n=M+1}^{\infty} c_n(S_N{}^\alpha)^n\right]\right| \leqslant \left\|\sum_{n=M+1}^{\infty} c_n(S_N{}^\alpha)^M\right\| \leqslant \sum_{n=M+1}^{\infty} |c_n|$$

as small as we wish uniformly in N. The same is true for the last term of (19), and the first term in (19) converges to zero as $N \to \infty$ by (14). By the same arguments, if $F(S_N{}^\alpha)$ and $G(S_N{}^\beta)$ are two functions defined by norm convergent power series, then

$$\lim_{N\to\infty} \rho_N(F(S_N{}^\alpha)G(S_N{}^\beta)) = F(S^\alpha)G(S^\beta) \quad (20)$$

In particular, since $S_N{}^0$ is bounded, the exponential (12) in $S_N{}^+(t)$ can be represented by its power series and

$$\lim_{N\to\infty} \rho_N(S_N{}^\alpha(t)) = S^\alpha(t) \quad (21)$$

The integrand $\Gamma_N(t_1) \cdots \Gamma_N(t_n)S_N(g)$ in the nth order term of

$$S_N(g, t) = \exp(iH_N t)\, S_N(g) \exp(-iH_N t) = T\left(\exp i \int_0^t \Gamma_N(t')dt'\right) S_N(g)$$

is a sum of products of $S_N{}^\alpha(t)$. Therefore, with (20) and (21) its average converges to the classical quantity $\Gamma(t_1) \cdots \Gamma(t_n)S(g)$. So do the time integrals. Finally, the whole series has the limit

$$\lim_{N\to\infty} \rho_N{}^t(S_N{}^\alpha(g)) = \lim_{N\to\infty} \rho_N(S_N{}^\alpha(g, t)) = S^\alpha(g, t) \quad (22)$$

since, as in (19), the series converges uniformly with respect to N. With the use of the same arguments, the result holds for arbitrary monomials of the $S_N{}^\alpha(g)$. ■

The proposition extends to other combinations of observables.

It is obviously true for multi-time correlations $\rho_N(\prod_r S_N{}^{\alpha_r}(g_r, t_r))$ that converge to their classical analogs $\prod_r S^{\alpha_r}(g_r, t_r)$. Later on, we shall also have to consider observables of the type

$$S_N^{\alpha\beta}(1, t) = \frac{1}{N}\sum_p \sigma_{pN}^{\alpha}(t)\sigma_p{}^\beta = \sum_\gamma \left[T\left(\exp i \int_0^t \Gamma_N(t')dt'\right)\right]^{\alpha\gamma} \frac{1}{N}\sum_p \sigma_p{}^\gamma \sigma_p{}^\beta \quad (23)$$

In view of the algebra of the Pauli matrices, $(1/N)\sum_p \sigma_p{}^\gamma \sigma_p{}^\beta$ is still an intensive observable of the form (6). Then it follows from Proposition 1 that asymptotically the classical propagator (18) can be used in (23).

2.4. Explicit Solution of the Semiclassical Equations of Motion

The simplest way of solving (15) is by the Laplace transform method. We set

$$\tilde{\sigma}_p{}^\alpha(z) = \int_0^\infty e^{-zt}\sigma_p{}^\alpha(t)\,dt, \qquad \mathrm{Re}\, z > 0$$

Then (15) becomes

$$\tilde{\sigma}_p{}^\alpha(z) = \frac{1}{z - 2i\alpha\epsilon}[\sigma_p{}^+ + i\alpha\mu S^+ \tilde{\sigma}_p{}^0(z - i\alpha\nu)], \qquad \alpha = +, - \tag{24}$$

$$z\tilde{\sigma}_p{}^0(z) - \sigma_p{}^0 = 2i\mu S^- \tilde{\sigma}_p{}^+(z + i\nu) - 2i\mu S^+ \tilde{\sigma}_p{}^-(z - i\nu) \tag{25}$$

Introducing (24) in (25) and solving for $\tilde{\sigma}_p{}^0(z)$, we get

$$\tilde{\sigma}_p{}^0(z) = \sum_\alpha \tilde{a}^\alpha(z)\sigma_p{}^\alpha \tag{26}$$

with

$$\begin{aligned} \tilde{a}^\alpha(z) &= \frac{2\mu S^\alpha(\mu S^0 + i\alpha z)}{z(z^2 + \omega^2)}, \qquad \alpha = +, - \\ \tilde{a}^0(z) &= \frac{z^2 + (\mu S^0)^2}{z(z^2 + \omega^2)} \end{aligned} \tag{27}$$

From (24) and (27) we obtain also

$$\tilde{\sigma}_p{}^+(z) = \sum_\alpha \tilde{b}^\alpha(z)\sigma_p{}^\alpha \tag{28}$$

with

$$\begin{aligned} \tilde{b}^+(z) &= \frac{1}{z - 2i\epsilon}[1 + i\mu S^+ a^+(z - i\nu)] \\ \tilde{b}^-(z) &= \frac{i\mu}{z - 2i\epsilon} S^- a^-(z - i\nu) \\ \tilde{b}^0(z) &= \frac{i\mu}{z - 2i\epsilon} S^+ a^0(z - i\nu) \end{aligned} \tag{29}$$

In (24) and (27), we have introduced the quantities

$$\nu = 2\epsilon + \mu S^0, \qquad \omega = \mu(S^{02} + 4S^+S^-)^{1/2} \tag{30}$$

which will be the characteristic frequencies of the motion. ω is proportional to the length of the total angular momentum and ν is the rotation frequency around the z axis.

We get the time dependence by inverting (26) and (28). Since the $\tilde{a}^\alpha(z)$ and $\tilde{b}^\alpha(z)$ are meromorphic functions of z, one has simply to evaluate the residues of the various poles. The result is

$$\sigma_p{}^0(t) = \sum_\alpha a^\alpha(t)\sigma_p{}^\alpha, \qquad \sigma_p{}^+(t) = \sum_\alpha b^\alpha(t)\sigma_p{}^\alpha, \qquad \sigma_p{}^-(t) = (\sigma_p{}^+(t))^* \tag{31}$$

$$a^\alpha(t) = \sum_\gamma e^{i\gamma\omega t} u_\gamma{}^\alpha, \qquad b^\alpha(t) = \sum_\gamma e^{i(\nu+\gamma\omega)t} v_\gamma{}^\alpha \tag{32}$$

with the following matrices of coefficients:

$$u_\gamma{}^\alpha = \begin{array}{c|ccc} \gamma \diagdown \alpha & + & - & 0 \\ \hline + & +\mu S^-(\omega - \mu S^0)/\omega^2 & -\mu S^+(\omega + \mu S^0)/\omega^2 & 2\mu^2 S^+ S^-/\omega^2 \\ - & -\mu S^-(\omega + \mu S^0)/\omega^2 & +\mu S^+(\omega - \mu S^0)/\omega^2 & 2\mu^2 S^+ S^-/\omega^2 \\ 0 & 2\mu^2 S^- S^0/\omega^2 & 2\mu^2 S^+ S^0/\omega^2 & (\mu S^0)^2/\omega^2 \end{array} \tag{33a}$$

$$v_\gamma{}^\alpha = \begin{array}{c|ccc} \gamma \diagdown \alpha & + & - & 0 \\ \hline + & (\omega - \mu S^0)^2/4\omega^2 & -(\mu S^+)^2/\omega^2 & +\mu S^+(\omega - \mu S^0)/2\omega^2 \\ - & (\omega + \mu S^0)^2/4\omega^2 & -(\mu S^+)^2/\omega^2 & -\mu S^+(\omega + \mu S^0)/2\omega^2 \\ 0 & 2\mu^2 S^+ S^-/\omega^2 & 2(\mu S^+)^2/\omega^2 & \mu^2 S^+ S^0/\omega^2 \end{array} \tag{33b}$$

One can note the following consistency relations:

$$\sum_\gamma u_\gamma{}^\alpha = \delta_{\alpha 0}, \qquad \sum_\gamma v_\gamma{}^\alpha = \delta_{\alpha +} \tag{34}$$

giving the correct initial values $\sigma_p{}^\alpha(t = 0) = \sigma_p{}^\alpha$, and

$$\sum_\alpha u_\gamma{}^\alpha S^\alpha = \delta_{\gamma 0} S^0, \qquad \sum_\alpha v_\gamma{}^\alpha S^\alpha = \delta_{\gamma 0} S^+$$

These identities ensure that the evolution law (16) of the intensive observables holds true.

Proposition 1 shows that for macroscopic states, the dynamics can be safely calculated in thermodynamic limit by formulas (32) and (33).

One should remark that this dynamics does not originate in general from the one-spin effective Hamiltonian

$$H_{\text{eff}} = \epsilon\sigma^0 - \mu(S^+\sigma^- + S^-\sigma^+) + \mu S^+S^- \tag{35}$$

H_{eff} is obtained by linearizing H_N around the classical C numbers S^+ and S^-, that is, setting in (3)

$$S_N^+S_N^- \simeq S^+S^- + (S_N^- - S^-)S^+ + (S_N^+ - S^+)S^-$$

It is known that H_{eff} is thermodynamically equivalent to H_N as $N \to \infty$. It has also been shown that H_{eff} produces asymptotically the correct dynamics if and only if the "gap equations"

$$S^+ = 0 \qquad \text{or} \qquad \nu = 2\epsilon + \mu S^0 = 0 \tag{36}$$

hold in the considered state.[12,14]

The same conclusion is immediately seen here, since when (36) holds, the semiclassical equations of motion (15) are precisely those generated by H_{eff}.

In particular, in an equilibrium state, (36) is valid ($S^+ = 0$ corresponds to the normal phase, $\nu = 0$ corresponds to the condensed phase) and time correlation functions can be evaluated using (35), as has been shown in Ref. 13.

In the next section we shall deal with nonequilibrium states and for this reason we need to consider the full time development as given by (32) and (33).

The Hamiltonian H_N is a special case of a general class of mean field systems that has been considered by Hepp and Lieb in Ref. 21. Such models have the feature that the equations of motion of intensive observables form a finite closed set of nonlinear differential equations, and this is the basic reason for their solvability. A simplification that occurs in our case is that this set, which is (11), can be explicitly solved, because of the conservation of pair number (S_N^0 is a constant of motion). Therefore, as $N \to \infty$, average values of $S_N^\alpha(t)$ follow trivially the classical equation (16) [cf. proof of (21)]. The same result is true in general for these models: average values of intensive observables obey the corresponding classical differential equations in macroscopic states[21] (cf. also the appendix of Ref. 14). Then the full microscopic dynamics is in principle completely determined by a propagator of the type (18), involving only the known classical trajectories of intensive observables.

3. DYNAMICS OF THE OPEN SYSTEM

3.1. Coupling with the Heat Bath

We consider now the BCS model described in Section 2.1 by the Hamiltonian (3) as an open system which can exchange energy with a heat bath.

The main effect of the bath will be to create or destroy electron pairs. We can think of it as being some external agency (as the lattices vibrations), which causes fluctuations of the number of Cooper pairs. It can also incorporate the effect of short-wavelength modes of the BCS system itself, which have already been thermalized and interact with the slowly varying observables such as the order parameter. In any case, we shall idealize the interaction mechanism with the heat bath in the simplest possible way. We assume:

(i) The coupling V is linear both in the electron pair operators $\sigma_p{}^+$ and $\sigma_p{}^-$ and in the creation and annihilation operators of excitations of the bath.

(ii) Relaxation of different p modes is not correlated through the bath, which we describe mathematically by attaching to each p an independent copy of the same given bath.

Thus, we write

$$\lambda V = \lambda \sum_p [\sigma_p{}^+ a_p(f) + \sigma_p{}^- a_p{}^+(f)] \tag{37}$$

$$a_p{}^+(f) = \int a_p{}^+(k) f(k)\, d^3k \tag{38}$$

$a_p{}^+(f)$ creates an excitation with wave function $f(k)$ in the p-bath associated with the mode p, and λ is some coupling constant.

Moreover, we take the evolution of the p-bath to be quasi-free, that is, generated by a quadratic Hamiltonian, which can formally be written as

$$H_p{}^B = \int d^3k\, \epsilon(k) a_p{}^+(k) a_p(k) \tag{39}$$

where $\epsilon(k)$ is the energy spectrum of the excitations.

It is equivalent to saying that the evolution of the bath is completely given in terms of that of the one-excitation states:

$$\exp(iH_p{}^B t)\, a_p(f) \exp(-iH_p{}^B t) = a_p(f_t) \tag{40}$$

with $f_t(k) = \{\exp[-i\epsilon(k)t]\} f(k)$.

Finally, we choose for mathematical convenience that the $a_p{}^+(k)$ and $a_p(k)$ obey the Fermi statistics anticommutation rules. Boundedness of Fermi operators renders mathematical proofs easier, but the same results would follow formally if we had Bose statistics instead.

Then the Hamiltonian for the total system is

$$H = H_N + \sum_p H_p{}^B + \lambda \sum_p [\sigma_p{}^+ a_p(f) + \sigma_p{}^- a_p{}^+(f)] \tag{41}$$

We are interested in finding the motion of a few intensive observables of the open BCS system. For this it is sufficient to deal with the part of the total state reduced to the BCS system by tracing out the degrees of freedom corresponding to the bath. This can be done with the Zwanzig projection technique[22] (see Ref. 23 for references), and the evolution of the reduced state is governed by the generalized master equation

$$\frac{d}{dt}\rho_t = i[H_N, \rho_t] + \lambda^2 \int_0^t ds\, \mathscr{K}(\lambda, s)\rho_{t-s} \tag{42}$$

Equation (42) is derived under the three following assumptions on the initial state of the coupled system:

(a) There are no correlations between the two systems at time $t = 0$.
(b) The initial state of the bath is stationary under its free evolution (40).
(c) The average value of the interaction vanishes in this state.

Assumptions (b) and (c) are obviously fulfilled for the interaction (37) if we have the bath in thermal equilibrium at time $t = 0$.

All the effects of the interaction with the bath are contained in the kernel $\mathscr{K}(\lambda, s)$ acting in the state space (density matrices) of the BCS system. $\mathscr{K}(\lambda, s)$ has a very complicated structure, involving in particular multitime correlation functions of the bath of all orders, and the integrodifferential equation (42) is only mathematically tractable in a limiting situation, the weak coupling or van Hove limit. For a discussion and a review of various applications of the weak coupling limit see Ref. 11. Let us recall here that in this limit, one lets $\lambda \to 0$ and simultaneously scales the observation time by setting $t = \lambda^{-2}\tau$, τ fixed. The mechanism of the weak coupling limit is immediately seen in (42) at an heuristic level. If we still denote by ρ_τ the state expressed as a function of the new parameter τ, then (42) becomes with this change of variable

$$\frac{d}{d\tau}\rho_\tau = \int_0^{\lambda^{-2}\tau} ds\, \mathscr{K}(\lambda, s)\rho_{\tau-\lambda^2 s} \tag{43}$$

(We have suppressed the free evolution part of (42), which plays no role when we study the evolution of a constant of the free motion of the BCS system; otherwise one removes it by going to the interaction picture.) Letting now formally $\lambda \to 0$ in (43), we get

$$\frac{d}{d\tau}\rho_\tau = G\rho_\tau, \qquad G = \int_0^\infty ds\, \mathscr{K}(0, s) \tag{44}$$

Thus, in the limit, the non-Markovian feature of (42) disappears, to give rise to an ordinary differential equation of semigroup type in the scaled time variable τ.

A correct derivation of (42) and a mathematical justification of the weak coupling limit are not trivial. We refer to Refs. 1 and 2 for a complete mathematical study. The essential property needed for the validity of the weak coupling limit is a sufficiently fast decrease of the time correlation functions of the bath in its initial state. Such decrease properties can only be obtained if the bath is treated as an infinitely extended quantum system. Therefore, the proper mathematical setting for the description of our bath is that of a quasi-free Fermi system with initial state satisfying the KMS equilibrium condition at inverse temperature β.[24] Since it is well known that quasi-free states are determined by their two-point correlation functions, the only correlation functions of the bath that are involved in the kernel $\mathscr{K}(\lambda, s)$ are

$$C_1(t) = \langle a_p(f_t)a_p^+(f)\rangle_\beta, \qquad C_2(t) = \langle a_p^+(f_t)a_p(f)\rangle_\beta \tag{45}$$

where $\langle\cdots\rangle_\beta$ are time-zero equilibrium averages. {For instance, an explicit expression of $C_1(t)$ with (38) and (39) is

$$C_1(t) = \left(\frac{1}{2\pi}\right)^3 \int d^3k \, \frac{|f(k)|^2 \exp[-i\epsilon(k)t]}{1 + \exp[-\beta\epsilon(k)]}\bigg\}$$

We assume that the decay in time of $C_1(t)$ and $C_2(t)$ is $O(1/t^{1+\eta})$, $\eta > 0$, for $t \to \infty$.

The KMS condition expressed in terms of the Fourier transforms $\tilde{C}_j(k) = \int dt \, e^{ikt} C_j(t)$, $j = 1, 2$, is

$$\tilde{C}_1(k) = e^{\beta k}\tilde{C}_2(-k) \tag{46}$$

Moreover, $\tilde{C}_1(k)$ and $\tilde{C}_2(k)$ are real nonnegative functions.

With this we are precisely in the range of application of theorem 2.3 of Ref. 1, asserting the existence of the weak coupling limit. (Remember that for finite N, states of the BCS system are in a finite-dimensional Hilbert space.)

Specializing to our case the general form of the generator G_N, (44), which can be found in Ref. 1, one obtains for any observable A_N of the finite BCS system

$$\begin{aligned} \frac{d}{d\tau}\rho_{N\tau}(A_N)|_{\tau=0} &= (G_N\rho_N)(A_N) \\ &= \rho_N\bigg[-\int_0^\infty dt \sum_p \{[A_N, \sigma_{pN}^+(t)]\sigma_p^- C_1(t) - \sigma_p^-[A_N, \sigma_{pN}^+(t)]C_2(t) \\ &\qquad + [A_N, \sigma_{pN}^-(t)]\sigma_p^+ C_2(t) - \sigma_p^+[A_N, \sigma_{pN}^-(t)]C_1(-t)\}\bigg] \end{aligned} \tag{47}$$

If A_N does not commute with H_N, one has to add to (47) the free evolution term of (42). [For a calculation very similar to (47), in the context of Ising systems, see Proposition 2 of Ref. 9.] Equation (47) is the basic formula upon which all our subsequent study of the open BCS system relies.

3.2. Equation of Motion of Intensive Observables

The master equation (47) is still a microscopic evolution law since it may be used to predict the value at time τ of any observable of the BCS system. In fact, we will be only interested in the macroscopic motion of the number of electron pairs $\frac{1}{2}(S_N{}^0 + 1)$ and the order parameter $S_N{}^+$ as $N \to \infty$. We therefore specialize A_N in (47) to be components of the vector S_N. Moreover, one must distinguish between rapidly and slowly varying observables (in the τ time scale). We set $R_N = S_N{}^+ S_N{}^-$ and we introduce the gap operator Δ_N (the modulus of the order parameter)

$$\Delta_N{}^2 = 4S_N{}^+ S_N{}^- = 4R_N \tag{48}$$

Clearly Δ_N and $S_N{}^0$, which are constants of the free BCS motion, evolve slowly when a weak coupling with the bath is switched on, whereas the free evolution (12) produces fast oscillations of the complex order parameter with frequency $\lambda^{-2}\nu$ ($\lambda \to 0$). We will be concerned here with the evaluation of (47) for Δ_N and $S_N{}^0$ in the thermodynamic limit, and we shall study the complex order parameter in Section 4.4.

It is more convenient to work for the moment with R_N instead of Δ_N. We shall come back to Δ_N later by the change of variable (48).

Let us denote by $F_R{}^N$ and $F_{S^0}^N$ the operators occurring in (47) for the choices $A_N = R_N$ and $A_N = S_N{}^0$, respectively, i.e.,

$$\begin{aligned} (G_N\rho_N)(R_N) &= \rho_N(F_R{}^N) = \langle F_R{}^N\rangle \\ (G_N\rho_N)(S_N{}^0) &= \rho_N(F_{S^0}^N) = \langle F_{S^0}^N\rangle \end{aligned} \tag{49}$$

$F_R{}^N$ and $F_{S^0}^N$ are the sums of four terms, the first of them involving

$$B_N(t) = \sum_p [R_N, \sigma_{pN}^+(t)]\sigma_p{}^- \tag{50}$$

for $F_R{}^N$, and

$$C_N(t) = \sum_p [S_N{}^0, \sigma_{pN}^+(t)]\sigma_p{}^- \tag{51}$$

for $F_{S^0}^N$.

All the other terms have a similar structure and it is sufficient to discuss (50) and (51). The commutators in (50) and (51) are straightforward to compute since both R_N and $S_N{}^0$ are constant of the free evolution. Thus, using equal-time commutation relations, we get

$$B_N(t) = \sum_p [S_N{}^+(t)S_N{}^-(t), \sigma_{pN}^+(t)]\sigma_p{}^- = -S_N{}^+(t)\frac{1}{N}\sum_p \sigma_{pN}^0(t)\sigma_p{}^- \quad (52)$$

$$C_N(t) = \sum_p [S_N{}^0(t), \sigma_{pN}^+(t)] = \frac{2}{N}\sum_p \sigma_{pN}^+(t)\sigma_p{}^- \quad (53)$$

Before evaluating (52) and (53) explicitly we note some points concerning the structure of $B_N(t)$ and $C_N(t)$.

(i) The only operators appearing in (52) and (53) are of the form (23), and products of such quantities with other intensive observables $S_N{}^\alpha(t)$. Therefore, in view of the nature of the propagator $T(\exp i\int_0^t \Gamma_N(t')\,dt')$, $B_N(t)$ and $C_N(t)$ can be expressed as functions of time-zero observables $S_N{}^\alpha$ only.

(ii) $B_N(t)$ and $C_N(t)$ commute with R_N and $S_N{}^0$. They commute with $S_N{}^0$ because of the gauge invariance of the free dynamics of the BCS model. Moreover, being functions of the $S_N{}^\alpha$ only, they commute with the length of the total "spin angular momentum"

$$(S_N{}^0)^2 + 2(S_N{}^+S_N{}^- + S_N{}^-S_N{}^+) = (S_N{}^0)^2 + 4R_N{}^2 - 2S_N{}^0/N = \omega_N{}^2/\mu^2$$

and consequently with R_N.

(iii) The preceding remarks imply that $B_N(t)$ and $C_N(t)$ can be expressed entirely in terms of the two mutually commuting observables R_N and $S_N{}^0$. Thus $F_R{}^N$ and F_{S0}^N can be considered as classical functions in the functional calculus of R_N and $S_N{}^0$ (or ω_N and $S_N{}^0$) (see Appendix C).

We assume from now on that the sequence of time-zero states ρ_N is macroscopic at S^α, $\alpha = +, -, 0$ [in the sense of the definition (14)], where S^α are arbitrarily prepared initial values.

Since, by remark (i), $B_N(t)$, $C_N(t)$, and the other similar quantities appearing in (47) involve only intensive observables for which Proposition 1 applies, the integrands of (47) converge as $N \to \infty$ if the sequence of states is macroscopic, and we can use the semiclassical form of the propagator (31). We find that the limits of these integrands are

$$\begin{aligned} &-S^+(t)[-a^0(t)S^- + \tfrac{1}{2}a^+(t)(1 + S^0)]C_1(t) \\ &+S^+(t)[a^0(t)S^- + \tfrac{1}{2}a^+(t)(1 - S^0)]C_2(-t) \qquad (54) \\ &+ \text{complex conjugate} \end{aligned}$$

if $A_N = S_N{}^+S_N{}^- = R_N$, and

$$\begin{aligned}&[-2b^0(t)S^- + 2b^+(t)\tfrac{1}{2}(1 + S^0)]C_1(t)\\ &- [2b^0(t)S^- + 2b^+(t)\tfrac{1}{2}(1 - S^0)]C_2(-t)\\ &+ \text{complex conjugate}\end{aligned} \tag{55}$$

if $A_N = S_N{}^0$.

As the sequence of the integrands is uniformly bounded with respect to N in operator norm by const $\times$ $[|C_1(t)| + |C_2(-t)| + |C_2(t)| + |C_1(-t)|]$, which is integrable by assumption, the dominated convergence theorem allows us to conclude that

$$\begin{aligned}\lim_{N\to\infty} \rho_N(F_R{}^N) = F_R(R, S^0) = &-S^+S^- \int_{-\infty}^{\infty} dt\, e^{ivt}a^0(t)C_1(t)\\ &+ \tfrac{1}{2}S^+(1 + S^0)\int_{-\infty}^{\infty} dt\, e^{ivt}a^+(t)C_1(t)\\ &- S^+S^-\int_{-\infty}^{\infty} dt\, e^{ivt}a^0(t)C_2(-t)\\ &- \tfrac{1}{2}S^+(1 - S^0)\int_{-\infty}^{\infty} dt\, e^{ivt}a^+(t)C_2(-t)\end{aligned} \tag{56}$$

and

$$\begin{aligned}\lim_{N\to\infty} \rho_N(F_{S^0}^N) = F_{S^0}(R, S^0) = &\ 2S^-\int_{-\infty}^{\infty} dt\, b^0(t)C_1(t)\\ &- (1 + S^0)\int_{-\infty}^{\infty} dt\, b^+(t)C_1(t)\\ &+ 2S^-\int_{-\infty}^{\infty} dt\, b^0(t)C_2(-t)\\ &+ (1 - S^0)\int_{-\infty}^{\infty} dt\, b^+(t)C_2(-t)\end{aligned} \tag{57}$$

In obtaining (56) and (57) we have used complex conjugate terms together with symmetry properties of the $a^\alpha(t)$ and $b^\alpha(t)$ to recover the full $(-\infty, \infty)$ time integration range.

Replacing now $b^\alpha(t)$ and $a^\alpha(t)$ by their explicit expressions (32), one sees that the integrals will give Fourier transforms of $C_1(t)$ and $C_2(t)$ at points ν, $\nu + \omega$, $\nu - \omega$. We can then use the KMS relation (46) to write the right-hand sides of (56) and (57) in the form

$$F_R(R, S^0) = \Big[\hat{C}(\nu)\frac{\mu^2 S^0}{\omega^2}\tanh\frac{\beta\nu}{2} - \hat{C}(\nu+\omega)\frac{\omega - \mu S^0}{2\omega^2}\left(\omega - \mu\tanh\frac{\beta(\nu+\omega)}{2}\right) - \hat{C}(\nu-\omega)\frac{\omega + \mu S^0}{2\omega^2}\left(\omega + \mu\tanh\frac{\beta(\nu-\omega)}{2}\right)\Big]R \tag{58}$$

and

$$F_{S^0}(R, S^0) = -\hat{C}(\nu)\frac{2\mu^2 R}{\omega^2}\tanh\frac{\beta\nu}{2} - \hat{C}(\nu+\omega)\frac{(\omega - \mu S^0)^2}{4\mu\omega^2}\left(\mu\tanh\frac{\beta(\nu+\omega)}{2} - \omega\right) - \hat{C}(\nu-\omega)\frac{(\omega + \mu S^0)^2}{4\mu\omega^2}\left(\mu\tanh\frac{\beta(\nu-\omega)}{2} + \omega\right) \tag{59}$$

with

$$\hat{C}(k) = \tilde{C}_1(k) + \tilde{C}_2(-k) \tag{60}$$

$$\omega = \mu[(S^0)^2 + 4R]^{1/2} = \mu[(S^0)^2 + \Delta^2]^{1/2}, \qquad R = \tfrac{1}{4}\Delta^2 \tag{61}$$

Equations (58) and (59) generate a nonlinear autonomous two-dimensional differential system for R and S^0:

$$\frac{d}{d\tau}R = F_R(R, S^0), \qquad \frac{d}{d\tau}S^0 = F_{S^0}(R, S^0)$$

Written in terms of Δ and S^0, this system becomes, with the change of variable (61),

$$\frac{d}{d\tau}\Delta = \frac{2}{\Delta}F_R(R, S^0) = F_\Delta(\Delta, S^0), \qquad \frac{d}{d\tau}S^0 = F_{S^0}(\Delta, S^0) \tag{62}$$

we shall abbreviate it by

$$\frac{d}{d\tau}X(\tau) = F(X(\tau)), \qquad X(\tau) = (\Delta(\tau), S^0(\tau)) \tag{63}$$

the components of the vector field $F(X)$ being defined by (62), (58), and (59).

Let us show that in the thermodynamic limit the macroscopic observables Δ_N and $S_N{}^0$ evolve indeed without fluctuations according to the deterministic equations (63). For this one has to find the behavior of higher order correlations $\rho_{N\tau}((S_N{}^0)^k(\Delta_N{}^2)^l)$ $(k, l = 0, 1, 2, \ldots)$ as $N \to \infty$.

Inserting, for instance, $A_N = (S_N{}^0)^k$ in (47), we find that (53) is replaced by

$$\frac{1}{N}\sum_p{}' [(S_N{}^0)^k, \sigma_{pN}^+(t)]\sigma_p{}^-$$
$$= \frac{1}{N}\sum_p [(S_N{}^0(t))^k, \sigma_{pN}^+(t)]\sigma_p{}^-$$
$$= \frac{1}{N}\sum_p \left\{\sum_{n=0}^{k-1} [S_N{}^0(t)]^n \frac{2}{N}\sigma_{pN}^+(t)[S_N{}^0(t)]^{k-n-1}\right\}\sigma_p{}^-$$
$$= k(S_N{}^0)^{k-1}\left[\frac{2}{N}\sum_p \sigma_{pN}^+(t)\sigma_p{}^-\right] + O\left(\frac{1}{N}\right) \tag{64}$$

where we have used the fact that equal-time commutators are always $O(1/N)$ uniformly in t. Thus, we find, as for (59),

$$\lim_{N\to\infty} (G_N\rho_N)(S_N{}^0)^k = F_{S^0}(X)k(S^0)^{k-1} = F_{S^0}(X)\frac{d}{dS^0}(S^0)^k$$

In the same way the commutator structure of (47) leads, for general monomials of Δ_N and $S_N{}^0$, to

$$\lim_{N\to\infty} (G_N\rho_N)[\Delta_N^{2l}(S_N{}^0)^k] = \left[F_\Delta(X)\frac{\partial}{\partial\Delta} + F_{S^0}(X)\frac{\partial}{\partial S^0}\right][\Delta^{2l}(S^0)^k] \tag{65}$$

This means that the generator of motion of any macroscopic probability distribution for $X = (\Delta, S^0)$ consists of the purely drift term (65) [in (65) we have the dual action of this drift term on observables]. Accordingly, the evolution of such probability distributions is induced by the two-dimensional flow generated by (63) and there are no fluctuations.

The fact that the open dynamics is described by a closed differential system involving only the two intensive observables Δ and S^0 is due to the mean field nature of the BCS Hamiltonian (3) and its gauge invariance. One should emphasize that (63) is valid far from equilibrium since initial values $X = (\Delta, S^0)$ can be any point in the physical domain that is the half-disk

$$\mathcal{D} = \{X: |X|^2 = (S^0)^2 + \Delta^2 \leqslant 1, \Delta \geqslant 0\} \tag{66}$$

The coupled evolution of Δ and S^0 results from an interaction with the bath that is not gauge invariant, thus modifying the number of Cooper pairs. If we had chosen instead of (37) a gauge-invariant interaction (like the scattering of electron pairs), S^0 would still be a constant of the motion of the open system, and (63) would reduce to a single equation for the gap parameter Δ.

The properties of the flow generated by (63) are the subject of the next section.

4. DESCRIPTION OF THE FLOW

4.1. Local Properties

It is well known that there will be locally a unique solution of our differential system (62) if the field satisfies a local Lipschitz condition. We show in Appendix C, Lemma 4, that if the correlation functions of the bath satisfy

$$\int_0^\infty |C_j(t)| t^k \, dt < \infty, \qquad j = 1, 2 \tag{67}$$

then $F(X)$ belongs to the class C^k of k-times continuously differentiable functions. In fact, if the Fourier transform $\hat{C}(k)$ is of class C^1, it is obvious in (58) and (59) that $F(X)$ is also C^1 for $\omega \neq 0$. But $\omega = 0$ is not a singular point. $F(X)$ can be defined by continuity at $\omega = 0$, setting

$$F(0) = \lim_{X \to 0} F(X) = (0, -\hat{C}(2\epsilon) \tanh \beta\epsilon) \tag{68}$$

and it can be checked by direct calculation that $F(X)$ is also C^1 at $\omega = 0$.

From now on, we assume that (67) holds for some $k \geqslant 1$, ensuring local existence and unicity of the flow in the disk $|X| \leqslant 1$.

Moreover, we assume also throughout this and the next subsection that $\hat{C}(k)$ is everywhere strictly positive. Cases where $\hat{C}(k)$ vanishes will be investigated in Section 4.3.

Before discussing the equilibria of the field, let us notice that the axis $\Delta = 0$ is an invariant manifold of the flow. Indeed for $\Delta = 0$, $\omega = \mu|S^0|$ and (62) reduces to

$$\Delta(\tau) = 0, \qquad \frac{d}{d\tau} S^0(\tau) = -\hat{C}(2\epsilon)[S^0(\tau) + \tanh \beta\epsilon] \tag{69}$$

(69) is simply the relaxation equation of a spin in an external magnetic field, as would be intuitively expected from the Hamiltonian (3).

The thermal equilibria of the free BCS model in the thermodynamic limit are given by the solutions of the mean field equations associated with H_{eff}, (35). They are

$$X_0 = (0, -\tanh \beta\epsilon) \tag{70}$$

which exists for all temperatures and corresponds to the normal phase.

If the temperature is less than the critical temperature T_c defined by[4]

$$\tanh \beta_c \epsilon = 2\epsilon/\mu, \qquad T_c = 1/k_B \beta_c \tag{71}$$

[4] We shall assume that the parameters ϵ and μ are such that $\epsilon > 0$, $\mu > 0$, and $2\epsilon/\mu < 1$.

there exists a second solution with $\Delta_s \neq 0$ corresponding to the superconducting phase

$$X_S = (\Delta_S, -2\epsilon/\mu) \tag{72}$$

where Δ_s is solution of the implicit equation

$$\tanh\left\{\beta\frac{\mu}{2}\left[\left(\frac{2\epsilon}{\mu}\right)^2 + \Delta_s^2\right]^{1/2}\right\} = \left[\left(\frac{2\epsilon}{\mu}\right)^2 + \Delta_s^2\right]^{1/2} \tag{73}$$

It is easily checked that X_0 and X_S are also equilibria of our vector field. It is of course expected that such thermal equilibrium values of the free BCS system are stationary under the evolution obtained in the weak coupling limit.

4.1.1. Normal Phase. Equation (69) shows that $F(X_0) = 0$ for all temperatures T. The linear part of $F(X)$ at X_0 is

$$(DF)(X_0) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

with

$$\begin{aligned} \lambda_1 &= -\frac{\hat{C}(2\epsilon - \mu \tanh \beta\epsilon)}{2 \tanh \beta\epsilon} \tanh\left[\frac{\beta}{2}(2\epsilon - \mu \tanh \beta\epsilon)\right] \\ \lambda_2 &= -\hat{C}(2\epsilon) < 0 \end{aligned} \tag{74}$$

Since for $T > T_c$, $\tanh \beta\epsilon < 2\epsilon/\mu$, both eigenvalues are negative, and X_0 is hyperbolic and asymptotically stable. The phrase portrait is a node. There is an exponential approach to equilibrium in both directions with relaxation times λ_1^{-1} and λ_2^{-1}.

For $T = T_c$, (71) shows that $\lambda_1 = 0$, and X_0 is no longer hyperbolic.

4.1.2. Superconducting Phase. For $T < T_c$, $\lambda_1 > 0$, the Δ direction becomes unstable and X_0 is a saddle point. One sees in (58) and (59) that a new equilibrium point occurs for $\nu = 0$ and $\mu \tanh(\beta\omega/2) = \omega$, which is precisely X_S.

Therefore, $T = T_c$ is a bifurcation point for the system (63) and the branches of location of equilibria are shown in Fig. 1.

The linear part of the field at $X = X_S$ is of the form

$$(DF)(X_S) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

One finds that

$$\begin{aligned} a + d = &-\frac{1}{4\omega^2}[\hat{C}(0)\beta\mu^3 \Delta^2 + \hat{C}(\omega)(\omega - \mu S^0)^2 \\ &+ \hat{C}(-\omega)(\omega + \mu S^0)^2]|_{X=X_S} < 0 \end{aligned}$$

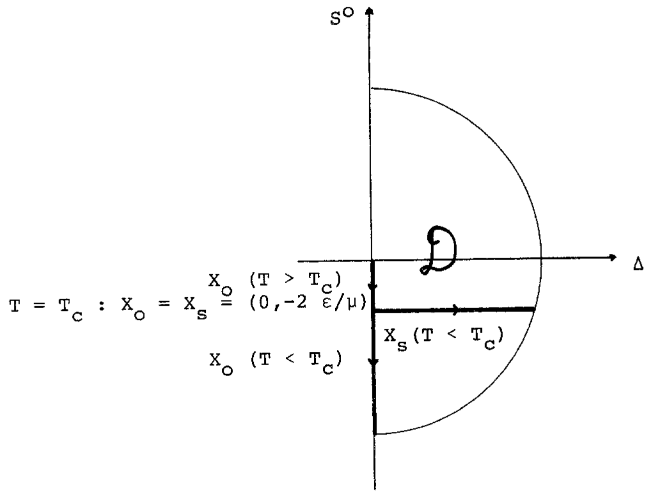

Fig. 1. Bifurcation of the set of equilibrium points.

and that $ad - bc$ is proportional (with a positive factor) to $(-\partial/\partial\Delta) \times \{\mu \tanh[\beta(\omega + \nu)/2] - \omega\}|_{X=X_S}$, which is seen to be a positive quantity for $T < T_c$. These facts ensure that both eigenvalues of $(DF)(X_S)$ have negative real parts for $T < T_c$. One can, moreover, prove that both eigenvalues are real; consequently the phase portrait is a node. Hence X_S is hyperbolic and asymptotically stable, and the approach to equilibrium is exponentially fast in a neighborhood of X_S. [Remember, however, that in view of (69) the trajectory with initial value $\Delta = 0$ converges always to X_0 as $\tau \to \infty$.]

Stability at the critical temperature cannot be decided here since in this case $X_0 = X_S$ is not hyperbolic. It will be discussed in the next section with the help of a Liapunov function. It will also be shown that X_0 for $T > T_c$ and X_0 and X_S for $T < T_c$ are the only equilibria of the field [when $\hat{C}(k)$ is strictly positive].

4.2. Global Properties

We begin with the proof of the existence of global solutions of Eq. (63). As usual, this follows from some a priori estimation on the solution. It is expected that, given an initial condition in the physical domain $\mathscr{D}$, (66), the solution that it defines remains in $\mathscr{D}$ as long as it exists. This is indeed true:

Lemma 1. $\mathscr{D}$ is positively invariant under the evolution determined by the field F.

Proof. We analyze the behavior of F on the boundary of $\mathscr{D}$. Direct computation gives

$$\left(X\cdot\frac{d}{d\tau}X\right)\bigg|_{\omega=\mu} = \tfrac{1}{4}\hat{C}(\nu+\mu)(1-S^0)^2\tanh\left(\beta\frac{\nu+\mu}{2}\right)-1$$

$$-\tfrac{1}{4}\hat{C}(\nu-\mu)(1+S^0)^2\left(\tanh\beta\frac{\nu-\mu}{2}+1\right) \qquad (75)$$

which is obviously negative since $|\tanh y| < 1$ and $\hat{C}(k) > 0$. This means that the field F points inward on the boundary of the unit disk, therefore preventing the trajectory from escaping from the disk. Moreover, each of the two half-disks $\Delta \geqslant 0$ and $\Delta \leqslant 0$ is separately invariant.

Indeed we know from (69) that the axis $\Delta = 0$ is itself a trajectory. As a consequence of local unicity, no trajectory with initial value $\Delta > 0$ can cross the line $\Delta = 0$. Thus $\mathscr{D}$ is positively invariant. In fact, the symmetry property of the field

$$F_\Delta(\Delta, S^0) = -F_\Delta(-\Delta, S^0), \qquad F_{S^0}(\Delta, S^0) = F_{S^0}(-\Delta, S^0) \qquad (76)$$

implies the symmetry of the phase portrait with respect to reflections around the Δ axis. ∎

Lemma 1 implies the existence of global solutions of (63).[25]

In order to be able to describe global properties of solutions, we shall build a Liapunov function. Let us recall that a Liapunov function relative to an equilibrium point $\bar{X}$ has to satisfy

$$\begin{aligned} &L(\bar{X}) = 0 \\ &L(X) > 0 \qquad \text{for} \quad X \neq \bar{X} \\ &\dot{L}(X) \leqslant 0 \end{aligned} \qquad (77)$$

If, furthermore, the last inequality is strict (except at point $\bar{X}$), L is a strict Liapunov function.

A motivation for the search of a Liapunov function is given by the remark that the free energy $\phi(\rho)$ considered as a functional of the state has to be minimum for equilibrium phases,

$$\phi(\rho) = U(\rho) - TS(\rho) \qquad (78)$$

where $U(\rho) = \operatorname{Tr} \rho H$ and $S(\rho) = -k_B \operatorname{Tr} \rho \ln \rho$ are the average energy and entropy.

The fact that $\phi(\rho_\tau)$ is a decreasing function of τ when ρ_τ evolves according to a dissipative semigroup obtained in the weak coupling limit is proved in Ref. 11. In Ref. 11, $-(d/d\tau)\phi(\rho_\tau)$ is identified with the entropy production and the result holds for quantum systems in finite-dimensional Hilbert spaces. The same property has also been used for the study of the stochastic Ising model in the thermodynamic limit.[26]

In our case we calculate the free energy associated with states that are macroscopic at S^α ($\alpha = +, -, 0$) in such a way that $\phi(\rho)$ becomes a function of the intensive observables only. This can be done simply with states that are products on all modes of the one-spin states (see Appendix A)

$$\rho = \begin{pmatrix} \frac{1}{2}(1 + S^0) & S^- \\ S^+ & \frac{1}{2}(1 - S^0) \end{pmatrix} \tag{79}$$

For such states and the Hamiltonian (3), the free energy density in the thermodynamic limit is

$$\phi(\rho) = \epsilon S^0 - \frac{\mu\Delta^2}{4} + (1/\beta)\operatorname{Tr}\rho\ln\rho$$

and with (79), setting $V(\Delta, S^0) = \beta\phi(\rho)$,

$$V(\Delta, S^0) = \beta\left(\epsilon S^0 - \frac{\mu\Delta^2}{4}\right) + \frac{1}{2}\left(1 + \frac{\omega}{\mu}\right)\ln\frac{1}{2}\left(1 + \frac{\omega}{\mu}\right) + \frac{1}{2}\left(1 - \frac{\omega}{\mu}\right)\ln\frac{1}{2}\left(1 - \frac{\omega}{\mu}\right) \tag{80}$$

Lemma 2. $\dot{V}(X) \leqslant 0$ in $\mathscr{D}$ and the only zeros of $\dot{V}(X)$ in $\mathscr{D}$ are X_0 for $T \geqslant T_c$, and X_0 and X_S for $T < T_c$.

Proof. By differentiating V and introducing (62), we obtain

$$\begin{aligned}\dot{V}(\Delta, S^0) &= \frac{d}{d\tau}V(\Delta, S^0) = F_\Delta(\Delta, S^0)\frac{\partial V}{\partial\Delta}(\Delta, S^0) + F_{S^0}(\Delta, S^0)\frac{\partial V}{\partial S^0}(\Delta, S^0) \\ &= -\hat{C}(\nu)\frac{\beta\mu^2}{2\omega^2}\Delta^2\nu\tanh\frac{\beta\nu}{2} - \hat{C}(\nu + \omega)\frac{(\omega - \mu S^0)^2}{8\mu\omega^2} \\ &\quad \times\left(\mu\tanh\frac{\beta(\nu + \omega)}{2} - \omega\right)\left[\beta(\nu + \omega) - \ln\frac{\mu + \omega}{\mu - \omega}\right] \\ &\quad - \hat{C}(\nu - \omega)\frac{(\omega + \mu S^0)^2}{8\mu\omega^2}\left(\mu\tanh\frac{\beta(\nu - \omega)}{2} + \omega\right) \\ &\quad \times\left[\beta(\nu - \omega) + \ln\frac{\mu + \omega}{\mu - \omega}\right]\end{aligned} \tag{81}$$

The first term is obviously negative. So is the second: The formula

$$\ln\frac{\mu + \omega}{\mu - \omega} = 2\operatorname{argth}\frac{\omega}{\mu} \tag{82}$$

shows that $\mu \tanh[\beta(\nu + \omega)/2] - \omega$ and $\beta(\nu + \omega) - \ln[(\mu + \omega)/(\mu - \omega)]$ have the same zeros and thus their product has a constant sign, which is positive. The same argument applies to the third term.

The above discussion not only proves the first part of the lemma, but also shows that the three terms in (81) have to vanish separately if $\dot{V}(X)$ vanishes, all of them being nonpositive. This can only be the case at $X = X_0$ for $T \geqslant T_c$ and X_0 and X_S for $T < T_c$ [notice that $\dot{V}(0, 0) = -\hat{C}(2\epsilon)\beta\epsilon \times \tanh \beta\epsilon \neq 0$]. ■

We define now

$$L_0(X) = V(X) - V(X_0), \qquad L_S(X) = V(X) - V(X_S) \tag{83}$$

and state:

Lemma 3. (i) For $T \geqslant T_c$, L_0 is a strict Liapunov function associated with X_0 on $\mathscr{D}$.

(ii) For $T < T_c$, L_S is a strict Liapunov function associated with X_S on $\mathscr{D} - \{X_0\}$.

Proof. The only thing to prove is the positivity of $L_0(X)$ and $L_S(X)$ in the appropriate range of temperature. This is done in Appendix B.

Proposition 2. (i) For $T \geqslant T_c$, X_0 is the only equilibrium point of $F(X)$; it is asymptotically stable and its basin of attraction is $\mathscr{D}$.

(ii) For $T < T_c$, X_0 and X_S are the only equilibrium points of $F(X)$; X_S is asymptotically stable and its basin of attraction is $\mathscr{D}' = \{X = (\Delta, S^0) \in \mathscr{D}: \Delta \neq 0\}$.

Proof. The nonexistence of equilibria different from X_0 and X_S is a direct consequence of Lemma 2, since an equilibrium has to be a zero of $\dot{V}(X)$.

Asymptotic stability of X_0 for $T \geqslant T_c$ and X_S for $T < T_c$ follows immediately from Lemma 3.

Thus, the only remaining question is the extent of the basins of attraction. We first treat the case $T \geqslant T_c$. Consider a trajectory associated with F and with an initial value in $\mathscr{D}$. Since by Lemma 1 this trajectory is confined in $\mathscr{D}$, which is a compact set, its ω-limit set Ω is nonempty. On the other hand, Ω is included into (Ref. 27, p. 539, Lemma 11.1)

$$\{X \in \mathscr{D}: \dot{V}(X) = 0\}$$

In this range of temperature this means by Lemma 2 that $\Omega = \{X_0\}$. Hence X_0 is the single ω-limit point of all trajectories in $\mathscr{D}$. Therefore, we can conclude that all trajectories tend to this point as τ goes to infinity (Ref. 27, p. 146, Corollary 1.1).

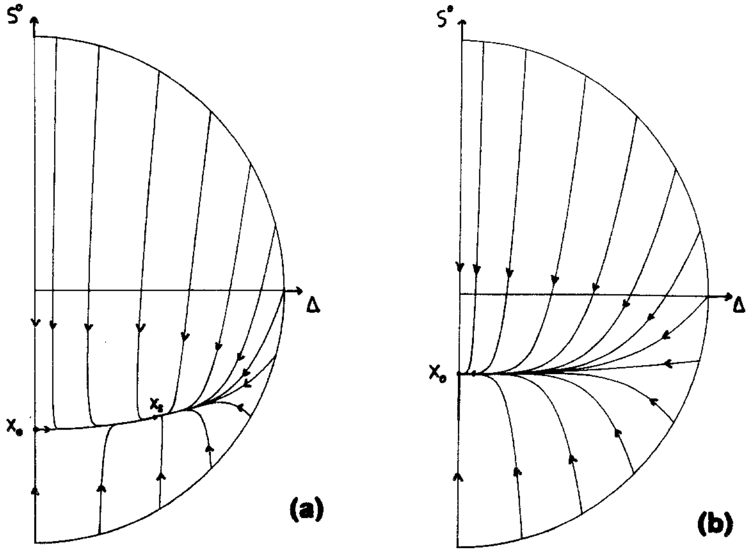

Fig. 2. Phase portrait of the field $F(X)$ at various temperatures. a), $T > T_c$ ($\beta = 1/3$); b), $T < T_c$ ($\beta = 0, 62$).

In the other case $T < T_c$, we obtain by a similar reasoning that Ω is included in $\{X_0, X_S\}$. The fact that the trajectory remains in the compact set $\mathscr{D}$ implies that Ω is connected (and nonempty) (Ref. 27, p. 145, Theorem 1.1). We thus have the two possible situations $\Omega = \{X_0\}$ or $\Omega = \{X_S\}$. For $T < T_c$, X_0 is a saddle point; hence there exist exactly two trajectories tending to X_0 as $\tau \to \infty$.[28] But we know these two trajectories from (69): They constitute the $\Delta = 0$ axis. Therefore, any other trajectory with initial $\Delta \neq 0$ must have X_S as a single ω-limit point and tends to it as $\tau \to \infty$. ∎

The proposition means that, except for the line $\Delta = 0$, when $T < T_c$, the thermodynamic phases are global attractors in the physical domain. The result holds also at the critical temperature, X_S being still asymptotically stable, although decay is not exponentially fast (critical slowing down).

A numerical study of the trajectories of system (3.26) provides an illustration of our previous analytic study.

The phase portraits drawn in Fig. 2 correspond to the following numerical values:

$$\hat{C} \equiv 7, \qquad \epsilon = 7, \qquad \mu = 4 \qquad \text{(which gives } \beta_C = 0, 549)$$

4.3. Structural Stability and Accidental Equilibria

We study here the influence on the dynamics of the coupling function $\hat{C}(k)$ with the bath. For finite systems, criteria have been given for the coupling with the bath causing the system to approach a unique equilibrium state under a dissipative semigroup evolution.[29,30] In our case, the analysis of Section 4.2 has shown that the thermodynamic points X_0 (for $T \geqslant T_c$) and X_S (for $T < T_c$) are global attractors of the flow provided that $\hat{C}(k) > 0$. If $\hat{C}(k)$ can vanish, there may occur other equilibria, which result from the nature of the coupling with the bath. The relevant mathematical concept that discriminates thermodynamic equilibria from others is the concept of structural stability. (See also Ref. 10.)

The vector field $F(X)$ is structurally stable if there exists a neighborhood of $F(X)$ (in a suitable topology on the set of vector fields) such that all fields in this neighborhood have topologically equivalent flows. This property is very desirable from the physical viewpoint. The dynamics of the system should not depend critically on the detailed nature of the thermal bath and of its interaction. If our field $F(X)$ is structurally stable, this precisely means that we can slightly modify the coupling function $\hat{C}(k)$ without changing qualitatively the phase portrait of trajectories. On the basis of the preceding analysis, we can also deduce the structural stability of $F(X)$ when $\hat{C}(k)$ is strictly positive (i.e., the only possible equilibria are the thermodynamic ones) and $T \neq T_c$.

For this we recall four results of Sections 4.1 and 4.2, obtained under the assumption of strict positivity of $\hat{C}(k)$:

(i) F points inward on the circle $|X| = 1$.
(ii) At any noncritical temperature the equilibria of F are hyperbolic.
(iii) No periodic orbit can exist, since we have a strict Liapunov function.
(iv) There is at most one saddle point. Hence no trajectory goes from saddle to saddle.

According to a theorem by Pontryagin and Andronov,[25] the four preceding facts guarantee the structural stability of F on $\mathscr{D}$ for $T \neq T_c$.

At $T = T_c$ the emergence of a nonhyperbolic equilibrium destroys structural stability. This has of course to be the case, since the phase portrait changes at the bifurcation point.

We drop now the restriction $\hat{C}(k) > 0$ and show that the new equilibria $\bar{X}$ ($\bar{X} \neq X_0, X_S$) that can possibly result of the zeros of $\hat{C}(k)$ are of very different nature from the thermodynamic ones: they are never hyperbolic, contrary to X_0 and X_S (for $T \neq T_c$). As a consequence, if such an equilibrium $\bar{X}$ occurs, the field cannot be structurally stable and $\bar{X}$ can be removed by a small perturbation of the function $\hat{C}(k)$. In this sense these new equilibria are produced by a peculiarity of the coupling with the bath and have to be considered as accidental.

We still suppose that $\hat{C}(k)$ is continuously differentiable, so that any zero of $\hat{C}(k)$ is also a zero of its derivative [remember that $\hat{C}(k) \geqslant 0$].

Let $\bar{X} \in \mathscr{D}$ ($\bar{X} \neq X_0, X_S$) be an equilibrium of $F(X)$ for which at least one of the quantities $\hat{C}(\nu)$, $\hat{C}(\nu + \omega)$, $\hat{C}(\nu - \omega)$ vanishes. Then $\dot{V}(X)$ must also vanish at $X = \bar{X}$. Using the expression (81) for $\dot{V}(X)$, which is a sum of three negative terms, we can explicitly enumerate all the ways of having $\dot{V}(X) = 0$. Then for each case, we compute the linearized part of the field at $\bar{X}$ and show that it has at least one eigenvalue that is zero. We shall not give the complete enumeration here, but content ourselves with a few illustrative cases.

Assume, for instance, that $\bar{X}$ is such that $\hat{C}(\nu + \omega) = 0$. Then, we can have $\dot{V}(\bar{X}) = 0$ with $\bar{X} \neq X_0, X_S$ if at least one of the following four situations occurs:

(a) $\Delta = 0$ and $(\omega + \mu S^0) = 0$.[5] We immediately see in (59) that $\bar{X}$ is a double zero of $F_{S^0}(X)$, thus showing that $\bar{X}$ is not hyperbolic.

(b) $\hat{C}(\nu) = 0$ and $\mu \tanh[\beta(\omega - \nu)/2] + \omega = 0$. Then one can check that the linearized part of the field at X is of the form $\begin{pmatrix} ab & ad \\ cb & cd \end{pmatrix}$. This matrix has obviously a zero eigenvalue.

(c) $\hat{C}(\nu) = 0$ and $(\omega + \mu S^0) = 0$. Then $\bar{X}$ is a double zero of both $F_{S^0}(X)$ and $F_\Delta(X)$.

(d) $\hat{C}(\nu) = 0$ and $\hat{C}(\omega - \nu) = 0$. Since the zeros of $\hat{C}(k)$ are themselves double, $\bar{X}$ is again nonhyperbolic.

If $\hat{C}(2\epsilon) = 0$ the origin is an equilibrium [cf. (68)]. By (69) all other points of the Δ axis are also equilibria. All of them are nonhyperbolic, since $F(X)$ is identically zero on this axis, and again the field cannot be structurally stable.

We summarize the situation in the following:

Proposition 3. Assume $\hat{C}(k) \in C^1$.

(i) If $\hat{C}(k) > 0$, the field is structurally stable on $\mathscr{D}$ for all $T \neq T_c$. It is not structurally stable for $T = T_c$.

(ii) If $\hat{C}(k) \geqslant 0$, and there exists an equilibrium $\bar{X}$ different from the thermodynamic ones X_0 and X_S, then the field is not structurally stable on $\mathscr{D}$.

This proposition provides a clear distinction between thermodynamic and accidental equilibria.

4.4. The Complex Order Parameter

We consider now the evolution of the three-dimensional vector S^X, S^Y, S^0 or equivalently S^+, S^-, S^0. If we set $A_N = S_N{}^+$ and work out the

[5] This situation can only occur if $\hat{C}(2\epsilon) = 0$; notice that then any point $(0, S^0)$, $S^0 < 0$, is an equilibrium.

expression (47) for a sequence of macroscopic states, we find that as $N \to \infty$, $S^+(\tau)$ obeys an equation of the form

$$\frac{d}{d\tau} S^+(\tau) = \left(\frac{F_\Delta(X(\tau))}{\Delta(\tau)} + i\lambda^{-2}\nu(\tau) + i\Phi(X(\tau))\right) S^+(\tau) \tag{84}$$

Equivalently, setting $S^+(\tau) = \frac{1}{2}\Delta(\tau)e^{i\varphi(\tau)}$, one gets for the phase of the order parameter

$$(d/d\tau)\varphi(\tau) = \lambda^{-2}\nu(\tau) + \Phi(X(\tau)) \tag{85}$$

In (84) and (85) we have included the phase $\lambda^{-2}\nu(\tau) = \lambda^{-2}[2\epsilon + \mu S^0(\tau)]$ of the free motion (which is of course of the order λ^{-2} in the τ time scale). In addition to it, there occurs another phase term $\Phi(X)$, which is due to the interaction with the bath. The explicit form of $\Phi(X)$, which can easily be found, is not of particular interest, and it is enough to note the following points.

(i) $\Phi(X)$ involves not only the functions $C_1(t)$ and $C_2(t)$, but also the functions $tC_1(t)$ and $tC_2(t)$. This is because the calculation of the commutators in (47) with the help of equal-time commutation relations now involves terms like

$$\begin{aligned}[\exp(i\mu S_N{}^0 t), \sigma_{pN}^+(t)] &= [\exp(2i\mu t/N) - 1]\sigma_{pN}^+(t)\exp(iS_N{}^0 t) \\ &\simeq 2(i\mu t/N)\sigma_{pN}^+(t)\exp(iS_N{}^0 t), \qquad N \to \infty\end{aligned}$$

Hence, for the existence of the infinite-volume limit one needs to assume here not only the integrability of $C_j(t)$ but also $\int |tC_j(t)|\, dt < \infty, j = 1, 2$.

(ii) $\Phi(X)$ is real and depends only on the gauge-invariant observables Δ and S^0.

(iii) $\Phi(X_0)$ and $\Phi(X_S)$ are in general different from zero for all temperatures.

For $T \geqslant T_c$ the complex order parameter tends to zero as $\tau \to \infty$. For $T < T_c$ we see from (85) that its phase behaves as

$$\varphi(\tau) \sim \Phi(X_S)\tau, \qquad \tau \to \infty \tag{86}$$

[remember that $\nu(\tau) \to 0$, $\tau \to \infty$, for $T < T_c$).

Therefore, $S^+(\tau)$ has asymptotically a circular motion with constant angular velocity $\Phi(X_S)$ on the circle of radius $\frac{1}{2}\Delta_S$, showing that this circle is an attracting orbit in the (S^X, S^Y) plane.

In fact, the three-dimensional differential system undergoes precisely a Hopf bifurcation in the (S^X, S^Y) plane: the origin becomes unstable at $T = T_c$ and gives rise to the attracting circle of radius $\frac{1}{2}\Delta_S$. One can check that the conditions for the Hopf bifurcation theorem are met.[31] In particular the derivative of the eigenvalue λ_1, (74), with respect to β,

$$\left.\frac{d\lambda_1}{d\beta}\right|_{\beta=\beta_c} = \frac{\mu^2\beta_c}{8\cosh^2\beta_c\epsilon}\hat{C}(0)$$

does not vanish at $T = T_c$ [if $\hat{C}(0) > 0$].

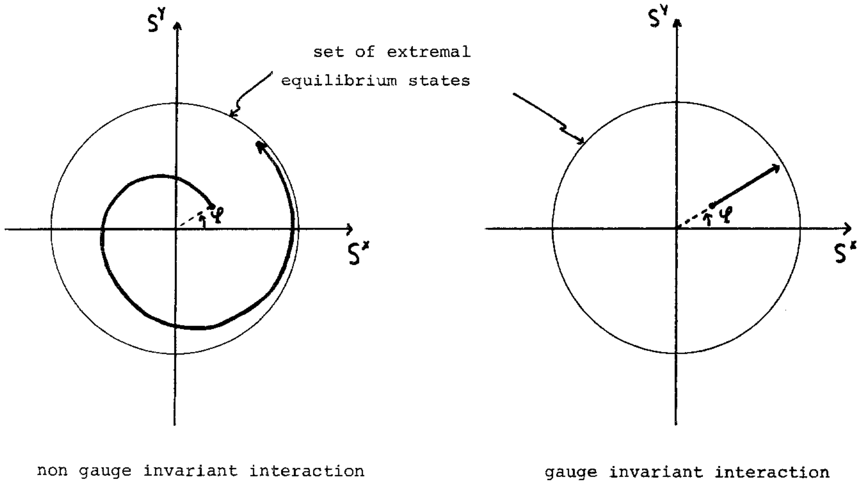

Fig. 3

Extremal equilibrium states of the BCS model can be labeled by the phase of S^+, according to the breaking of gauge invariance. Suppose that we choose a non-gauge-invariant initial state with

$$\lim_{N\to\infty} \rho_N(S_N{}^+) = S^+ = \tfrac{1}{2}\Delta e^{i\varphi} \neq 0$$

for $T < T_c$ we have asymptotically as $\tau \to \infty$

$$S^+(\tau) \simeq \tfrac{1}{2}\Delta_S \exp\{i[\varphi + \Phi(X_S)\tau]\} \neq 0$$

This shows that a non-gauge-invariant initial state is not driven by the dissipative semigroup (47) to a steady state, but goes over the set of extremal equilibrium states with constant angular velocity (Fig. 3). This feature is due to the fact that the interaction with the bath is itself non-gauge-invariant. One can check that the phase of the order parameter remains constant in time if the interaction is gauge-invariant (like the scattering of Cooper pairs). In the latter case, the symmetry breaking is obtained dynamically by the choice of non-gauge-invariant initial conditions.

5. FLUCTUATIONS

5.1. Fluctuation Observables

We know that intensive observables in macroscopic states evolve without fluctuations in the thermodynamic limit according to the deterministic equations (63) (see Section 3.2). In order to exhibit the fluctuations, we must treat the dynamics not to the leading order in N, but to the order $\sqrt{N}$. It is indeed a general feature of macroscopic bodies that, with the exception of critical points, the mean square deviations of intensive observables are $O(1/N)$ as $N \to \infty$. We are thus led to define the fluctuation observable of $S_N(g)$ around the mean value $\rho_N(S_N(g)) = \langle S_N(g) \rangle$ by

$$\hat{S}_N(g) = \sqrt{N}[S_N(g) - \langle S_N(g) \rangle] \tag{87}$$

Such fluctuation observables are $O(1)$ in states with normal fluctuations. If ρ_N is a macroscopic sequence of states at $S(g)$ [definition (14)], we say that ρ_N has normal fluctuations if $\lim_{N \to \infty} \rho_N(\Pi_r \hat{S}_N^{\alpha_r}(g_r))$ exists for all monomials of fluctuation observables $\hat{S}_N^{\alpha_r}(g_r)$, $g_r \in C^0(\Omega)$. (For a discussion of states with normal fluctuation see Ref. 21 and Appendix A.)

Since the commutator of fluctuation observables does not vanish as $N \to \infty$, the evolution of fluctuations is in general a truly quantum process and cannot be described in terms of a classical probability distribution. We shall not deal here with this full quantum process, but restrict our attention to the subset $\hat{X}_N = (\hat{\Delta}_N, \hat{S}_N^0)$ of two mutually commuting observables. In this commutative case one gets a classical description in view of the following remark. If the limit of averages of all monomials of $\hat{\Delta}_N$ and $\hat{S}_N^0$ exists, the limit exists also for all continuous functions $f(\hat{X}_N)$ with compact support, since such functions can be approximated uniformly by polynomials. Hence

$$\lim_{N \to \infty} \rho_N(f(\hat{X}_N)) = \hat{\rho}(f) \tag{88}$$

defines a bounded linear functional $\hat{\rho}$ on $C_0^0(\mathbb{R}^2)$. By the Riesz theorem, $\hat{\rho}$ yields a measure $d\hat{\rho}(\hat{X})$, $\hat{X} \in \mathbb{R}^2$, which is the joint probability measure for the fluctuations $\hat{X} = (\hat{\Delta}, \hat{S}^0)$ in the thermodynamic limit.

To study the dynamics of fluctuations around the trajectories, we introduce the fluctuation observables at time τ:

$$\hat{X}_N(\tau) = (\hat{\Delta}_N(\tau), \hat{S}_N^0(\tau))$$

$$\hat{\Delta}_N(\tau) = \sqrt{N}(\Delta_N - \langle \Delta_N \rangle_\tau), \qquad \hat{S}_N^0(\tau) = \sqrt{N}(S_N^0 - \langle S_N^0 \rangle_\tau) \tag{89}$$

where $\langle \Delta_N \rangle_\tau = \rho_{N\tau}(\Delta_N)$, $\langle S_N^0 \rangle_\tau = \rho_{N\tau}(S_N^0)$. Let $\rho_{N\tau} = (\exp G_N \tau)\rho_N$ evolve with the semigroup (47). $\rho_{N\tau}$ has normal fluctuations at time τ if

$$\lim_{N \to \infty} \rho_{N\tau}[f(\hat{X}_N(\tau))] = \hat{\rho}_\tau(f) \tag{90}$$

exists for all $f \in C^0(\mathbb{R}^2)$.

This limit again defines a measure $d\hat{\rho}_\tau(\hat{X})$ on $\mathbb{R}^2$ which is clearly the probability measure for fluctuations around $X(\tau)$ at time τ. The evolution of fluctuations still has the semigroup property:

Let $\hat{\rho}_{\tau_2,\tau_1}$ denote the distribution of fluctuations at time τ_2, given the initial distribution $\hat{\rho}_{\tau_1}$. By the definition (90)

$$\hat{\rho}_{\tau_2,\tau_1}(f) = \lim_{N \to \infty} [(\exp G_N \tau_2)\rho_{N\tau_1}][f(\hat{X}_N(\tau_2, \tau_1))]$$

Here $\hat{X}_N(\tau_2, \tau_1)$ is the fluctuation observable at time τ_2 associated with the initial state $\rho_{N\tau_1}$. Since $\rho_{N\tau}$ evolves itself according a semigroup, we obtain

$$\hat{\rho}_{\tau_2,\tau_1} = \hat{\rho}_{\tau_2+\tau_1} \tag{91}$$

In order to find the evolution of $\hat{\rho}_\tau$ it is therefore sufficient to calculate its generator (for a differentiable function f)

$$\begin{aligned}\frac{d}{d\tau}\hat{\rho}_\tau(f)|_{\tau=0} &= \lim_{N\to\infty}\frac{d}{d\tau}\rho_{N\tau}[f(\hat{X}_N(\tau))]|_{\tau=0} \\ &= \lim_{N\to\infty}\{(G_N\rho_N)[f(\hat{X}_N)] \\ &\quad - \sqrt{N}[\langle F_\Delta{}^N\rangle\langle\partial_1 f(\hat{X}_N)\rangle + \langle F^N_{S_0}\rangle\langle\partial_2 f(\hat{X}_N)\rangle]\}\end{aligned} \tag{92}$$

We prove in the next section that if ρ_N is a sequence of macroscopic states at X with normal fluctuations, the limit (92) exists and is given by

$$\frac{d}{d\tau}\hat{\rho}_\tau(f)|_{\tau=0} = \hat{\rho}\left(\left\{[(DF)(X)\hat{X}]\cdot\frac{\partial}{\partial\hat{X}} + \left[K(X)\frac{\partial}{\partial\hat{X}}\right]\cdot\frac{\partial}{\partial\hat{X}}\right\}f\right) \tag{93}$$

where $(DF)(X)$ is the linear part of the field at X and

$$K(X) = \begin{pmatrix} K_{\Delta\Delta}(X) & K_{\Delta S^0}(X) \\ K_{S^0\Delta}(X) & K_{S^0S^0}(X)\end{pmatrix}$$

is a 2×2 symmetric matrix depending on the point X.

If $d\hat{\rho}_\tau(\hat{X}) = \mu_\tau(\hat{X})\,d\hat{\Delta}\,d\hat{S}^0$ has a differentiable probability density $\mu_\tau(\hat{X})$, (93) together with the semigroup law (91) is equivalent to the linear Fokker–Planck equation

$$\frac{\partial}{d\tau}\mu_\tau(\hat{X}) = \left\{-\frac{\partial}{\partial\hat{X}}\cdot\{(DF)[X(\tau)]\hat{X}\} + \left[K(X(\tau))\frac{\partial}{\partial\hat{X}}\right]\cdot\frac{\partial}{\partial\hat{X}}\right\}\mu_\tau(\hat{X}) \tag{94}$$

where $(DF)[X(\tau)]$ and the diffusion matrix $K(X(\tau))$ are evaluated here at the point $X(\tau)$ of the trajectory defined by the initial condition X.

Equation (94) expresses our main result concerning the dynamics of fluctuations: in the thermodynamic limit the averages of the fluctuation observables follow the linearized equation of motion (63) around the trajectory $X(\tau)$ and their probability distribution is Gaussian.

In fact, (94) is the Fokker–Planck equation associated with the stochastic differential equation

$$\frac{d}{d\tau}\hat{X}(\tau) = (DF)[X(\tau)]\hat{X}(\tau) + \varphi(\tau) \tag{95}$$

where $\varphi(\tau)$ is a Gaussian Markovian (but nonstationary) random force defined by

$$\langle \varphi_\Delta(\tau) \rangle = \langle \varphi_{S^0}(\tau) \rangle = 0$$

and

$$\begin{aligned} \langle \varphi_\Delta(\tau_1)\varphi_\Delta(\tau_2) \rangle &= 2K_{\Delta\Delta}(X(\tau_1))\, \delta(\tau_1 - \tau_2) \\ \langle \varphi_\Delta(\tau_1)\varphi_{S^0}(\tau_2) \rangle &= 2K_{\Delta S^0}(X(\tau_1))\, \delta(\tau_1 - \tau_2) \\ \langle \varphi_{S^0}(\tau_1)\varphi_{S^0}(\tau_2) \rangle &= 2K_{S^0 S^0}(X(\tau_1))\, \delta(\tau_1 - \tau_2) \end{aligned} \tag{96}$$

However, one should note that a complete proof that the fluctuations are given by the stochastic process (95) is not obtainable in the framework of the master equation. To prove that the full process is Gaussian and Markovian would require in addition to the Fokker–Planck equation (94) a knowledge of the set of all multitime correlations $\langle \varphi(\tau_1)\varphi(\tau_2) \cdots \varphi(\tau_n) \rangle$. Such information is not available from the master equation (47), which gives the state only at the single time τ.

5.2. Generator of the Fokker–Planck Equation

This section is devoted to the proof that the limit (92) exists for states with normal fluctuations and is the Fokker–Planck generator (93). To simplify the presentation it is convenient to use instead of $\hat{\Delta}_N$ the fluctuation observable

$$\hat{R}_N = \sqrt{N}(R_N - \langle R_N \rangle)$$

with, as before,

$$\begin{aligned} R_N &= S_N{}^+ S_N{}^- = \tfrac{1}{4}\Delta_N{}^2, \qquad R = \tfrac{1}{4}\Delta^2 \\ F_R(Y) &= dR/d\tau = \tfrac{1}{2}\Delta F_\Delta(X) \end{aligned} \tag{97}$$

We set

$$\begin{aligned} Y_N &= (R_N, S_N{}^0), \qquad Y = (R, S^0) \\ \hat{Y}_N &= (\hat{R}_N, \hat{S}_N{}^0), \qquad \hat{Y} = (\hat{R}, \hat{S}^0) \end{aligned} \tag{98}$$

We shall come back later to the $\hat{\Delta}$ variable by a simple change of variable.

It is sufficient to evaluate (92) for monomials $f(\hat{Y}_N) = \hat{R}_N{}^k(\hat{S}_N{}^0)^l$. We shall treat explicitly only the case $\hat{R}_N{}^k$. Once the mechanism is revealed, the reader can easily supplement the arguments and the calculations needed for the general case.

As a first step we establish an asymptotic development of $(G_N\rho_N)(\hat{R}_N{}^k)$

with leading term $O(\sqrt{N})$. Recalling the form (47) of G_N, we see that it is a sum of four terms, the first of them involving

$$\sum_p [\hat{R}_N{}^k, \sigma_{pN}^+(t)]\sigma_p{}^- \tag{99}$$

We obtain for (99) the following asymptotic development written in terms of the fluctuation observables:

$$\sum_p [\hat{R}_N{}^k, \sigma_{pN}^+(t)]\sigma_p{}^- = \sqrt{N}\, k\hat{R}_N^{k-1}B_N(t) + k(k-1)\hat{R}_N^{k-2}D_N(t) + O(1/\sqrt{N}) \tag{100}$$

with

$$B_N(t) = \sum_p [R_N, \sigma_{pN}^+(t)]\sigma_p{}^- = -S_N{}^+(t)\frac{1}{N}\sum_p \sigma_{pN}^0(t)\sigma_p{}^- \tag{101}$$

$$D_N(t) = \frac{1}{2}\sum_p [R_N, \sigma_{pN}^+(t)]\sigma_p{}^0 S_N{}^- = -\frac{1}{2}S_N{}^+(t)\frac{1}{N}\sum_p \sigma_{pN}^0(t)\sigma_p{}^0 S_N{}^- \tag{102}$$

Before proving (100) we note that the remarks (i)–(iii) of Section 3.2 apply to $B_N(t)$ and $D_N(t)$. The $B_N(t)$ and $D_N(t)$ commute with R_N and $S_N{}^0$, and can be considered as functions of $Y_N = Y + \hat{Y}_N/\sqrt{N}$, or equivalently, of $\hat{Y}_N$ only. The right-hand side of (100) can therefore be understood as a C-function in the functional calculus of $\hat{Y}_N$. In this sense, $O(1/\sqrt{N})$ means a function of the fluctuation observables that is bounded by some polynomial $(1/\sqrt{N})\sum_{kl} b_{kl}|\hat{R}_N|^k|\hat{S}_N{}^0|^l$.

The proof of the formula (100) is by induction on k. We start with the following commutator identity:

$$\begin{aligned}\sum_p [\hat{R}_N^{k+1}, \sigma_{pN}^+(t)]\sigma_p{}^- &= \hat{R}_N\sum_p [\hat{R}_N{}^k, \sigma_{pN}^+(t)]\sigma_p{}^- \\ &\quad + \sum_p [\hat{R}_N, \sigma_{pN}^+(t)]\sigma_p{}^-\hat{R}_N{}^k + \sum_p [\hat{R}_N, \sigma_{pN}^+(t)][\hat{R}_N{}^k, \sigma_p{}^-]\end{aligned} \tag{103}$$

The second term on the right of (103) is simply $\sqrt{N}\,\hat{R}_N{}^k B_N(t)$.

We can work out $[\hat{R}_N{}^k, \sigma_p{}^-]$ into a multiple commutator expansion. Since any commutator brings a factor $1/\sqrt{N}$, the leading term in this expansion is

$$k[\hat{R}_N, \sigma_p{}^-]\hat{R}_N^{k-1} = (1/\sqrt{N})\sigma_p{}^0 S_N{}^- k\hat{R}_N^{k-1} \tag{104}$$

the rest being also a polynomial in $\hat{R}_N$. With (104) we see that the last term of (103) is

$$2k\hat{R}_N^{k-1}D_N(t) + O(1/\sqrt{N})$$

Taking this and (100) into account in (103) proves (100) by induction.

The operator $B_N(t)$ which occurs in the first term of (100) is exactly the quantity (52) that we had to consider to compute the vector field F in Section 3.2. Once we have obtained developments similar to (100) for all terms of (47), we can write

$$(G_N\rho_N)(\hat{R}_N{}^k) = \sqrt{N}\, k\rho_N(\hat{R}_N^{k-1}F_R{}^N) + k(k-1)\rho_N(\hat{R}_N^{k-2}K_{RR}^N) + O(1/\sqrt{N}) \tag{105}$$

The terms $O(\sqrt{N})$ add up to form the R component $F_R{}^N$ of the field (for finite N), and the contributions $O(1)$ give the new contribution K_{RR}^N.

In estimating the remainder, we use the fact that it is bounded by a polynomial of fluctuation observables. Hence $\rho_N(|\hat{R}_N|^k|\hat{S}_N{}^0|^l) \leqslant [\rho_N(\hat{R}_N^{2k}(\hat{S}_N{}^0)^{2l})]^{1/2}$ remains bounded as $N \to \infty$ if ρ_N has normal fluctuations.

The first term in (105), together with the second term in (92), gives

$$k\sqrt{N}[\langle\hat{R}_N^{k-1}F_R{}^N\rangle - \langle\hat{R}_N^{k-1}\rangle\langle F_R{}^N\rangle] \tag{106}$$

We introduce now the operator $F_R(Y_N)$, which is obtained by substituting in the field $F_R(Y)$ the operators $\frac{1}{2}(S_N{}^+S_N{}^- + S_N{}^-S_N{}^+) = R_N - (1/2N)S_N{}^0$ and $S_N{}^0$ in place of the numbers R and S^0. The operator $F_R(Y_N)$ is a well-defined operator in the functional calculus of the commuting observables R_N and $S_N{}^0$.

It is proved in Appendix C that if the correlation functions of the bath $C_1(t)$ and $C_2(t)$ satisfy an exponential decay condition, the following estimate holds:

$$F_R{}^N - F_R(Y_N) = O(1/N) \tag{107}$$

This allows us to write (106) as

$$k\sqrt{N}[\langle\hat{R}_N^{k-1}F_R(Y_N)\rangle - \langle\hat{R}_N^{k-1}\rangle\langle F_R(Y_N)\rangle] + O(1/\sqrt{N}) \tag{108}$$

Using now the relation between intensive observables and fluctuation observables

$$Y_N = \langle Y_N\rangle + (1/\sqrt{N})\hat{Y}_N$$

and the differentiability of $F_R(Y)$ (Lemma 4 of Appendix C), we have a limited Taylor expansion

$$\sqrt{N}\, F_R\left(\langle Y_N\rangle + \frac{1}{\sqrt{N}}\hat{Y}_N\right) = \sqrt{N}\, F_R(\langle Y_N\rangle) + \frac{\partial F_R}{\partial R}(\langle Y_N\rangle)\hat{R}_N + \frac{\partial F_R}{\partial S^0}(\langle Y_N\rangle)\hat{S}_N{}^0 + O\left(\frac{1}{\sqrt{N}}\right) \tag{109}$$

Inserting (109) in (108) and taking into account that by definition $\langle \hat{R}_N \rangle = \langle \hat{S}_N{}^0 \rangle = 0$, we find that (108) becomes

$$\frac{\partial F_R}{\partial R}(\langle Y_N \rangle) k \langle \hat{R}_N{}^k \rangle + \frac{\partial F_R}{\partial S^0}(\langle Y_N \rangle) k \langle \hat{R}_N^{k-1} \hat{S}_N{}^0 \rangle + O\left(\frac{1}{\sqrt{N}}\right) \tag{110}$$

The treatment of the second term in (105) is similar: We introduce the operator $K_{RR}(Y_N)$ associated with the function $K_{RR}(Y) = \lim_{N\to\infty} \langle K_{RR}^N \rangle$ and have the estimate

$$K_{RR}^N - K_{RR}(Y_N) = O(1/N)$$

The regularity of K_{RR} (obtained by similar arguments as for F_R) gives

$$K_{RR}(Y_N) = K_{RR}(\langle Y_N \rangle) + O(1/\sqrt{N}) \tag{111}$$

In the limit $N \to \infty$, $\langle Y_N \rangle$ converges to Y and we obtain from (92), (105), (110), and (111)

$$\begin{aligned} \frac{d}{d\tau} \hat{\rho}_\tau(\hat{R}^k)|_{\tau=0} &= \frac{\partial F_R(Y)}{\partial R} k \hat{\rho}(\hat{R}^k) + \frac{\partial F_R(Y)}{\partial S^0} k \hat{\rho}(\hat{R}^{k-1}\hat{S}^0) \\ &\quad + k(k-1) K_{RR}(Y) \hat{\rho}(\hat{R}^{k-2}) \end{aligned} \tag{112}$$

Equation (112) is precisely (93) written in terms of the variable $\hat{Y}$ and for the special choice $f(\hat{Y}) = \hat{R}^k$.

Finally, we come back to the $\hat{X}$ variable, noting that

$$\hat{R}_N = \tfrac{1}{4}\sqrt{N}(\Delta_N{}^2 - \langle \Delta_N \rangle^2) = \tfrac{1}{2}\langle \Delta_N \rangle \hat{\Delta}_N + O(1/\sqrt{N}) \tag{113}$$

Thus in the thermodynamic limit we can set

$$\hat{R} = \tfrac{1}{2}\Delta\hat{\Delta} \tag{114}$$

and

$$\begin{aligned} \frac{d}{d\tau} \hat{\rho}_\tau(\hat{R}^k)|_{\tau=0} &= \lim_{N\to\infty} \frac{d}{d\tau} \rho_{N\tau}(\hat{R}_N{}^k(\tau))|_{\tau=0} \\ &= \frac{1}{2} F_\Delta(X) \left(\frac{\Delta}{2}\right)^{k-1} \hat{\rho}(\hat{\Delta}^k) + \left(\frac{\Delta}{2}\right)^k \frac{d}{d\tau} \hat{\rho}_\tau(\hat{\Delta}^k)|_{\tau=0} \end{aligned} \tag{115}$$

Transforming (112) according to (114) and (115) gives now exactly (93) with $K_{\Delta\Delta}(X) = (4/\Delta^2) K_{RR}(Y)$.

As an example we give the explicit form of the diffusion coefficient $K_{\Delta\Delta}(X)$ of the gap variable Δ. Using the semiclassical propagator again, we find from (102) and the other contributions to K_{RR}^N

$$\begin{aligned} K_{\Delta\Delta}(X) &= \frac{\mu^2 \hat{C}(\nu)(S^0)^2}{2\omega^2} + \hat{C}(\nu+\omega) \frac{\mu\Delta^2}{4\omega^2} \left(\mu - \omega \tanh \beta \frac{\nu+\omega}{2}\right) \\ &\quad + \hat{C}(\nu-\omega) \frac{\mu\Delta^2}{4\omega^2} \left(\mu - \omega \tanh \beta \frac{\omega-\nu}{2}\right) \geqslant 0 \end{aligned} \tag{116}$$

5.3. Asymptotic Behavior of Fluctuations

We conclude this section by a brief discussion of the behavior of fluctuations (we treat only the case where X_0 and X_S are the only equilibria).

The asymptotic behavior of the mean square fluctuations $\langle\hat{\Delta}^2\rangle_\tau$, $\langle(\hat{S}^0)^2\rangle_\tau$, and $\langle\hat{\Delta}\hat{S}^0\rangle_\tau$ differs according to the values of the temperature and the initial value X of the trajectory $X(\tau)$. (Remember that by definition $\langle\hat{\Delta}\rangle_\tau = \langle\hat{S}^0\rangle_\tau = 0$.)

1. We consider first the case where $T \neq T_c$ and X belongs to the basin of attraction of X_0 (for $T > T_c$) or X_S (for $T < T_c$).

We can replace asymptotically $X(\tau)$ by X_e ($X_e = X_0$, $T > T_c$; or $X_e = X_S$, $T < T_c$) in (94) and we get

$$\frac{d}{d\tau}\begin{pmatrix} \langle\hat{\Delta}^2\rangle_\tau \\ \langle(\hat{S}^0)^2\rangle_\tau \\ \langle\hat{\Delta}\hat{S}^0\rangle_\tau \end{pmatrix} = \begin{pmatrix} 2\dfrac{\partial F_\Delta}{\partial\Delta}(X_e) & 0 & 2\dfrac{\partial F_\Delta}{\partial S^0}(X_e) \\ 0 & 2\dfrac{\partial F_{S^0}}{\partial S^0}(X_e) & 2\dfrac{\partial F_{S^0}}{\partial\Delta}(X_e) \\ \dfrac{\partial F_{S^0}}{\partial\Delta}(X_e) & \dfrac{\partial F_\Delta}{\partial S^0}(X_e) & \dfrac{\partial F_\Delta}{\partial\Delta}(X_e) + \dfrac{\partial F_{S^0}}{\partial S^0}(X_e) \end{pmatrix}$$

$$\times \begin{pmatrix} \langle\hat{\Delta}^2\rangle_\tau \\ \langle(\hat{S}^0)^2\rangle_\tau \\ \langle\hat{\Delta}\hat{S}^0\rangle_\tau \end{pmatrix} + \begin{pmatrix} K_{\Delta\Delta}(X_e) \\ K_{S^0S^0}(X_e) \\ K_{\Delta S^0}(X_e) \end{pmatrix} \tag{117}$$

The eigenvalues of the above matrix are $2\lambda_1$, $2\lambda_2$, and $\lambda_1 + \lambda_2$, where λ_1 and λ_2 are the eigenvalues of $(DF)(X_e)$ (for $T > T_c$ the matrix is diagonal). We know that λ_1 and λ_2 have negative real parts (see Section 4.1). Therefore, fluctuations are driven exponentially fast to their thermal equilibrium value. This is the phenomenon of regression of fluctuations.

2. If $T < T_c$ and the initial value of Δ is zero, $X(\tau)$ tends to the unstable equilibrium X_0. In this range of temperature $\lambda_1 = (\partial F_\Delta/\partial\Delta)(X_0)$ is positive [see Eqs. (74)], and (117) gives rise to an exponential blowup of $\langle\hat{\Delta}^2\rangle_\tau$. This amplification of fluctuations characterizes the instability of the axis $\Delta = 0$. One can say heuristically that even small fluctuations suffice to destabilize the system and create a tendency for the system to get to the stable ordered phase X_S.

3. If $T = T_c$ is critical, then X_0 is no longer hyperbolic (although asymptotically stable) and we have to expand $F_\Delta(X)$ to higher order in Δ to find the rate of approach to equilibrium. In the domain of validity of the time-dependent Ginzburg–Landau theory [$\Delta \simeq 0$, $T \simeq T_c$, see Eq. (124),

Section 6], we have

$$\frac{d}{d\tau}\Delta(\tau) = -d\Delta^3(\tau) \tag{118}$$

$$\frac{d}{d\tau}\langle\hat{\Delta}^2\rangle_\tau = -6d\Delta^2(\tau)\langle\hat{\Delta}^2\rangle_\tau + K_{\Delta\Delta}(X_0) \tag{119}$$

Equation (118) gives the asymptotic behavior

$$\Delta(\tau) = O(\tau^{-1/2}) \tag{120}$$

This is the typical slowing down that occurs at the critical temperature.

We see in (116) that $K_{\Delta\Delta}(X_0) = K_{\Delta\Delta}(0, -2\epsilon/\mu) = \frac{1}{2}\hat{C}(0) > 0$. Since by (120), $\Delta^2(\tau) = O(\tau^{-1})$, we deduce from (119) that $\langle\hat{\Delta}^2\rangle_\tau$ diverges like $O(\tau)$ as $\tau \to \infty$. This divergence is the dynamical manifestation of the fact (well known from the equilibrium theory) that the fluctuations of the order parameter are no longer normally distributed at the critical point.

6. COMPARISON WITH PHENOMENOLOGICAL DYNAMICAL THEORIES

Since the dynamics of our open BCS model can be exhibited explicitly and without approximations starting from Hamiltonian mechanics (though in the weak coupling limit), it is interesting to compare it with the usual phenomenological dynamical theories. In the thermodynamics of irreversible processes[32] the time rate of change of the order parameter Δ is assumed to be given by

$$\frac{d}{d\tau}\Delta = -\mathscr{H}\frac{d}{d\Delta}\phi(\Delta), \qquad \mathscr{H} > 0 \tag{121}$$

where $\mathscr{H}$ is a kinetic coefficient and $\phi(\Delta)$ is a suitable thermodynamic potential. When $\phi(\Delta)$ takes the Ginzburg–Landau form, (121) is referred to as the time-dependent Ginzburg–Landau theory.[33] In a general situation, $\Delta = \Delta(q)$ is space dependent and the right-hand side of (121) has to be understood in the sense of functional derivative. Since our BCS model is strictly mean field, there is clearly in our case no spatial structure and Δ is the only relevant order parameter. For the sake of comparison, let us calculate the Ginzburg–Landau potential $\phi(\Delta)$ corresponding to the model. $\phi(\Delta)$ is defined as the free energy associated with the effective Hamiltonian (35), considered now as a function of Δ; we have

$$\phi(\Delta) = -\frac{1}{\beta}\ln \operatorname{Tr}\exp(-\beta H_{\text{eff}}) = \mu\frac{\Delta^2}{4} - \frac{1}{\beta}\ln 2\cosh\frac{\beta\omega(\Delta)}{2} \tag{122}$$

with $\omega(\Delta) = [(2\epsilon)^2 + \mu^2\Delta^2]^{1/2}$. The $\phi(\Delta)$ is of course minimum for $\Delta = 0$ or $\Delta = \Delta_S$.

The dynamics defined by (121) and (122) is therefore

$$\frac{d}{d\tau}\Delta = -\mathscr{H}\frac{d}{d\Delta}\phi(\Delta) = -\frac{\mathscr{H}\mu\Delta}{2}\left(1 - \frac{\mu\tanh[\beta\omega(\Delta)/2]}{\omega(\Delta)}\right) \tag{123}$$

If we investigate the motion of $\Delta(\tau)$ in the neighborhood of the critical temperature, we find that $\Delta(\tau)$ is very small and it makes sense to expand (123) in the form

$$d\Delta/d\tau = a(\beta)\Delta - d\Delta^3 \tag{124}$$

where the coefficient d is taken at $\beta = \beta_c$.

We find from (123)

$$\begin{aligned} a(\beta) &= \frac{\mathscr{H}\mu}{4\epsilon}(\mu\tanh\beta\epsilon - 2\epsilon)\begin{cases} > 0, & T < T_c \\ < 0, & T > T_c \end{cases} \\ d &= -\frac{\mathscr{H}\mu^3}{16\epsilon^2}\left(\frac{\beta_c\mu}{2\cosh^2\beta_c\epsilon} - 1\right) > 0 \end{aligned} \tag{125}$$

Equation (124) is the usual simple phenomenological equation describing the bifurcation of the order parameter. Equations (123) and (124) should be now compared with our differential system (62).

We remark that in (62) the two variables S^0 and Δ evolve in a coupled way: the rate of change of $\Delta(\tau)$ depends on the instantaneous value of $S^0(\tau)$. Moreover, the vector field is not a gradient system, since it can be easily checked that in general $(\partial/\partial S^0)F_\Delta \neq (\partial/\partial\Delta)F_{S^0}$. There does not exist a general thermodynamic potential depending on S^0 and Δ that gives rise to (62).

However, we can simplify our system in the neighborhood of the critical temperature. If T is sufficiently close to T_c, we see in (74) that $\lambda_1 \ll \lambda_2$; this means that $S^0(\tau)$ is a much more rapidly relaxing variable than $\Delta(\tau)$. In this case $S^0(\tau)$ is a "stable mode" compared to $\Delta(\tau)$ and could be eliminated from the equations by invoking the adiabatic elimination principle (Ref. 19, Chapter 7). We shall make an even cruder approximation by assuming that for $T \simeq T_c$, $S^0(\tau)$ has already been driven to its equilibrium value and therefore remains constant in the course of time. When we set $S^0(\tau) = -2\epsilon/\mu$ in our equation, we get

$$\frac{d}{d\tau}\Delta = -\left[\hat{C}(\omega(\Delta))\frac{\omega(\Delta) + 2\epsilon}{2\mu\omega(\Delta)} + \hat{C}(-\omega(\Delta))\frac{\omega(\Delta) - 2\epsilon}{2\mu\omega(\Delta)}\right]\frac{d}{d\Delta}\phi(\Delta) \tag{126}$$

Equation (126) is not exactly identical with (123), but remembering that it holds only for $T \simeq T_c$ and small Δ, we can expand it around $\Delta = 0$ in the same manner as for (124). Noticing that $\omega(\Delta) - 2\epsilon \simeq \frac{1}{2}(\mu\Delta)^2$, only the first term of the brackets in (126) contributes and we find that this expansion coincides precisely up to Δ^3 with (124), provided that we identify the kinetic coefficient $\mathscr{K}$ with $2\hat{C}(2\epsilon)/\mu$.

Thus, we conclude that our equations reduce to the time dependent Ginzburg–Landau form (124) for T close to T_c and small Δ. In other ranges of temperatures, especially if the relaxation times λ_1^{-1} and λ_2^{-1} are of the same order of magnitude and if initial values are far from equilibrium, one has to keep the full differential system (62).

APPENDIX A

In this appendix we show that the set of states of the BCS model with uncorrelated p modes (i.e., independent spin states) are macroscopic and have normal fluctuations. This is a simple noncommutative version of the central limit theorem (in this context see Ref. 34). Consider the product states

$$\rho_N = \bigotimes_{p\epsilon\Omega} \rho_p$$

with single spin states

$$\rho_p = \begin{pmatrix} \frac{1}{2}[1 + S^0(p)] & S^+(p) \\ S^-(p) & \frac{1}{2}[1 - S^0(p)] \end{pmatrix} \tag{A1}$$

The $S^\alpha(p)$ are given functions in $C^0(\Omega)$, $|S^\alpha(p)| \leqslant 1$. We first prove that these states are macroscopic; we have

$$\rho_N\left(\prod_{i=1}^n S_N^{\alpha_i}(g_i)\right) = \frac{1}{N^n} \sum_{p_1,\dots,p_n} g_1(p_1)\cdots g_n(p_n)\langle \sigma_{p_1}^{\alpha_1}\cdots\sigma_{p_n}^{\alpha_n}\rangle$$

We separate the above sum into two parts: a sum on $p_1,\dots, p_n$ with p_i two by two distinct, and the remainder. In the remainder each sum contains at most N^{n-1} terms. Thus, according to the boundedness of the summand the remainder is $O(1/N)$ and vanishes in the thermodynamic limit. The nonvanishing term is

$$\frac{1}{N} \sum_{\substack{p_1,\dots,p_n \\ p_i \text{ distinct}}} \prod_{i=1}^n g_i(p_i)\langle\sigma_{p_i}^{\alpha_i}\rangle = \prod_{i=1}^n \left[\frac{1}{N}\sum_p g_i(p) S^{\alpha_i}(p)\right] + O\left(\frac{1}{N}\right)$$

which converges to the monomial $\Pi_{i=1}^n S^{\alpha_i}(g_i)$ as $N \to \infty$.

In order to show that ρ_N has normal fluctuation, we calculate the limits of averages of monomials of fluctuation observables.

$$\langle \hat{S}_N^{\alpha_1}(g_1)\cdots\hat{S}_N^{\alpha_n}(g_n)\rangle = \frac{1}{N^{n/2}} \sum_{p_1,\dots,p_n} g_1(p_1)\cdots g_n(p_n)\langle\tau_{p_1}^{\alpha_1}\cdots\tau_{p_n}^{\alpha_n}\rangle \tag{A2}$$

with $\tau_p{}^\alpha = \sigma_p{}^\alpha - \langle\sigma_p{}^\alpha\rangle = \sigma_p{}^\alpha - S_{(p)}^\alpha$; we have $\langle\tau_p{}^\alpha\rangle = 0$.

Let $\mathscr{P}(n,j) = \bigcup_{\mu=1}^{j} \mathscr{C}_\mu{}^n$ be a partition of $\{1, 2,..., n\}$ into j subsets $\mathscr{C}_\mu{}^n$. Each partition determines a subset $(q_1,..., q_j)_{\mathscr{P}}$ of the n indices $p_1,..., p_n$ by identifying the p_k with $k \in \mathscr{C}^n$, $\mu = 1, 2,..., j$. We can decompose the sum (A2) into

$$\sum_{p_1,\ldots,p_n} = \sum_{j=1}^{n} \sum_{\mathscr{P}(n,j)} \sum_{\substack{(q_1,\ldots,q_j)\mathscr{P} \\ q_i \text{ distinct}}} \tag{A3}$$

where the second sum is on all partitions $\mathscr{P}(n,j)$ and the third sum runs for each partition on the associated set of indices $q_1 \in \Omega,..., q_j \in \Omega$, $q_1 \neq q_2 \neq \cdots \neq q_j$. We distinguish the following cases in (A3):

(a) $j < n/2$. The third sum in (A3) has N^j bounded terms. Their contribution to (A2) is $O(N^{j-n/2})$ and thus vanishes as $N \to \infty$.

(b) $j > n/2$. Now $\mathscr{P}(n,j)$ has at least a one-element subset. Then $\langle \tau_{p_1}^{\alpha_1} \cdots \tau_{p_n}^{\alpha_n} \rangle$ has certainly a factor $\langle \tau_{p_j}^{\alpha_j} \rangle$ which is zero.

And if n is even:

(c) $j = n/2$ and $\mathscr{P}(n, n/2)$ has a one-element subset. These terms are zero as in (b).

(d) $j = n/2$ and $\mathscr{P}(n, n/2)$ is a partition of $\{1, 2,..., n\}$ in $n/2$ pairs.

The sum of all such contributions can be written, up to terms $O(1/N)$, as

$$\sum_{\rho} \prod_{k=1}^{n/2} \frac{1}{N} \sum_{p} g_{\rho(2k-1)}(p) g_{\rho(2k)}(p) \langle \tau_p^{\alpha_{\rho(2k-1)}} \tau_p^{\alpha_{\rho(2k)}} \rangle$$

$$= \sum_{\rho} \prod_{k=1}^{n/2} \langle \hat{S}_N^{\alpha_{\rho(2k-1)}}(g_{\rho(2k-1)}) \hat{S}_N^{\alpha_{\rho(2k)}}(g_{(\rho 2k)}) \rangle \tag{A4}$$

The sum runs on all partitions, of $\{1, 2,..., n\}$ in pairs, i.e., ρ is a permutation of $1, 2,..., n$ such that $\rho(2k-1) < \rho(2k)$ and $\rho(2k-1) < \rho(2k+1)$.

If n is odd, cases (c) and (d) do not occur, and (A2) converges to zero as $N \to \infty$. If n is even, the only nonvanishing contribution, given by (A4), involves correlations $\langle \hat{S}_N^{\alpha_1}(g_1) \hat{S}_N^{\alpha_2}(g_2) \rangle$ of order two, which converge as $N \to \infty$. For instance,

$$\lim_{N\to\infty} \langle \hat{S}^0(g_1) \hat{S}^0(g_2) \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_p g_1(p) g_2(p) \langle [\sigma_p{}^0 - S^0(p)]^2 \rangle$$

$$= \frac{1}{|\Omega|} \int_\Omega d^3p\, g_1(p) g_2(p) [1 - S^{02}(p)]$$

We see that for n even the correlation of order n, (A2), converges to the sum of the $(n-1)!!$ products of correlations of order two. Except for the non-commutativity of fluctuation observables, this structure characterizes Gaussian averages.

APPENDIX B. POSITIVITY OF $L_0(x)$ AND $L_s(x)$

We first show the following facts:

(a) If $T \geqslant T_c$, $V(X)$ has the single stationary point X_0 (which is a local minimum when $T > T_c$).

(b) If $T < T_c$, $V(X)$ has two stationary points X_0 and X_S; X_0 is a saddle point, whereas X_S is a local minimum.

The partial derivatives of $V(X)$ are

$$\begin{aligned} \frac{\partial V(X)}{\partial \Delta} &= -\frac{1}{2}\beta\mu\Delta + \frac{\mu\Delta}{2\omega}\ln\frac{\mu+\omega}{\mu-\omega} \\ \frac{\partial V(X)}{\partial S^0} &= \beta\epsilon + \mu\frac{S^0}{2\omega}\ln\frac{\mu+\omega}{\mu-\omega} \end{aligned} \tag{B1}$$

Notice that these derivatives are continuous at the origin, with $(\partial V/\partial\Delta)|_{X=0} = 0$ and $(\partial V/\partial S^0)|_{X=0} = \beta\epsilon$. Therefore $V(X)$ is C^1 in the open disk $|X| < 1$.

Recalling that $\frac{1}{2}\ln[(1+Y)/(1-Y)] = \operatorname{argth} Y$, one verifies immediately that X_0 ($T \geqslant T_c$) and X_0 and X_S ($T < T_c$) are zeros of (B1). Conversely, $(\partial/\partial\Delta)V(X)$ vanishes if $\omega = 0$, or $\Delta = 0$ (with $S^0 \neq 0$), or $\omega = \omega_S$. We have that $\omega = 0$ is excluded.[6] In the two other cases, $(\partial/\partial S^0)V(X)$ vanishes if $S^0 = -\tanh\beta\epsilon$ or $S^0 = -2\epsilon/\mu$, giving X_0 and X_S.

In order to establish that X_0 and X_S are local minima of $V(X)$ in the appropriate range of temperature, we have to show that the second-order partial derivative matrix is positive definite.

We find

$$\begin{aligned} \left.\frac{\partial^2 V}{\partial\Delta^2}\right|_{X_0} &= \frac{\beta}{\tanh\beta\epsilon}\left(\epsilon - \frac{\mu}{2}\tanh\beta\epsilon\right) \\ \left.\frac{\partial^2 V}{\partial(S^0)^2}\right|_{X_0} &= \frac{1}{1-\tanh^2\beta\epsilon} > 0 \\ \left.\frac{\partial^2 V}{\partial\Delta\,\partial S^0}\right|_{X_0} &= 0 \end{aligned} \tag{B2}$$

Since $(\partial^2 V/\partial\Delta^2)|_{X_0}$ has the sign of $\beta_c - \beta$, X_0 is a local minimum of V for $T > T_c$, and a saddle point for $T < T_c$.

[6] Because $(\partial/\partial S^0)V(X)$ does not vanish at the origin.

We similarly have

$$\begin{aligned} \left.\frac{\partial^2 V}{\partial \Delta^2}\right|_{X_S} &= \mu^2 \Delta_S{}^2 \gamma \\ \left.\frac{\partial^2 V}{\partial (S^0)^2}\right|_{X_S} &= 4\epsilon^2 \gamma + \beta \frac{\mu}{2} \\ \left.\frac{\partial^2 V}{\partial \Delta\, \partial S^0}\right|_{X_S} &= -2\mu\epsilon \Delta_S \gamma \end{aligned} \tag{B3}$$

where we have set

$$\gamma = \frac{\mu^2}{\omega_S{}^2}\left(\frac{1}{\mu^2 - \omega_S{}^2} - \frac{\beta}{2\mu}\right) \tag{B4}$$

The sign of γ is found by the following remark:

$$\gamma = \frac{\mu^2}{\omega_S{}^2} \frac{d}{d\omega}\left(\frac{1}{\mu} \operatorname{argth} \frac{\omega}{\mu} - \frac{\beta\omega}{2\mu}\right)\Big|_{\omega = \omega_S}$$

By definition of ω_S, we know that the function

$$\frac{1}{\mu} \operatorname{argth} \frac{\omega}{\mu} - \beta \frac{\omega}{2\mu} \tag{B5}$$

is zero at $\omega = \omega_S$. One easily checks that, with $\beta > \beta_c$, (B5) is negative for $0 < \omega < \omega_S$ and positive for $\omega > \omega_S$, thus implying $\gamma \geqslant 0$; moreover, one can exclude $\gamma = 0$, since the derivative of (B5) has only one positive root, which has to be less than ω_S because (B5) is zero at the origin. Therefore $\gamma > 0$ for $T < T_c$.

Thus the diagonal elements of the partial derivative matrix and its determinant $\beta\mu^3 \Delta_S{}^2\gamma/2$ are positive, proving (b).

Lemma 3 is proved if we know that X_0 (for $T \geqslant T_c$) and X_S (for $T < T_c$) are absolute minima of $V(X)$ in $\mathscr{D}$. Since $V(X)$ is continuous on the closed disk $|X| \leqslant 1$, $V(X)$ reaches its absolute minimum either inside the disk or on its boundary $|X| = 1$. If $V(X)$ takes its minimum inside the disk, this must be at X_0 for $T \geqslant T_c$, by (a), or at X_S for $T < T_c$, by (b). Therefore, the proof is complete if we show

$$\begin{aligned} V(X)|_{|X|=1} &> V(X_0) = -\ln 2 \cosh \beta\epsilon, && T \geqslant T_c \\ V(X)|_{|X|=1} &> V(X_S) = \frac{\beta\mu}{4} \Delta_S{}^2 - \ln 2 \cosh \frac{\beta\omega_S}{2} && T < T_c \end{aligned} \tag{B6}$$

In fact $V(X)|_{|X|=1} = \beta\epsilon S^0 - \frac{1}{4}\mu[1 - (S^0)^2]$ is a function of the single variable S^0, which takes its minimum value $-\beta(\epsilon^2/\mu + \frac{1}{4}\mu)$ at $S^0 = -2\epsilon/\mu$.

Therefore, to have (B6) it is enough to show

$$h(\epsilon) \equiv \ln 2 \cosh \beta\epsilon - \frac{\beta\epsilon^2}{\mu} - \frac{\beta\mu}{4} > 0, \qquad T \geqslant T_c \tag{B7}$$

$$g(\Delta)|_{\Delta=\Delta_s} \equiv \ln 2 \cosh \frac{\beta\mu}{2}\left[\left(\frac{2\epsilon}{\mu}\right)^2 + \Delta_S{}^2\right]^{1/2} - \frac{\beta\mu\Delta_S{}^2}{4} - \frac{\beta\epsilon^2}{\mu} - \frac{\beta\mu}{4} > 0, \qquad T < T_c \tag{B8}$$

For a fixed $\beta \leqslant \beta_c$, consider the function $h(\epsilon)$ of ϵ defined by (B7) in the interval $\epsilon_0 \leqslant \epsilon \leqslant \mu/2$ (with $\tanh \beta\epsilon_0 = 2\epsilon_0/\mu$). Since $(d/d\epsilon)h(\epsilon) = \beta(\tanh \beta\epsilon - 2\epsilon/\mu) \leqslant 0$ in this interval, $h(\epsilon)$ is nonincreasing. Therefore, $h(\epsilon) \geqslant h(\mu/2) = \ln(1 + e^{-\beta\mu}) > 0$.

For a fixed $\beta > \beta_c$, consider the function $g(\Delta)$ of Δ defined by (B8) in the interval $\Delta_S \leqslant \Delta \leqslant \Delta_1$. The value $\Delta_1 = [1 - (2\epsilon/\mu)^2]^{1/2}$ is the largest possible equilibrium value of Δ. The function $g(\Delta)$ is decreasing in this interval since

$$\frac{d}{d\Delta} g(\Delta) = \frac{\beta\Delta}{2[(2\epsilon/\mu)^2 + \Delta^2]^{1/2}}\left\{\mu \tanh\left(\frac{\beta\mu}{2}\left[\left(\frac{2\epsilon}{\mu}\right)^2 + \Delta^2\right]^{1/2}\right) - \mu\left[\left(\frac{2\epsilon}{\mu}\right)^2 + \Delta^2\right]^{1/2}\right\} \leqslant 0$$

for $\Delta \geqslant \Delta_S$. Hence $g(\Delta_S) \geqslant g(\Delta_1) = \ln 2\cosh(\beta\mu/2) - \frac{1}{2}\beta\mu = \ln(1 + e^{-\beta\mu}) > 0$.

APPENDIX C

We study in more detail the operator $F_R{}^N$ and its relation with the operator $F_R(Y_N)$.

Lemma 4. If $\int_0^\infty |C_j(t)| t^k \, dt < \infty$, $F(X)$ belongs to C^k.

Proof. We see in (32) and (33) that $a^\alpha(t)$ and $b^\alpha(t)$ ($\alpha = +, -, 0$) are C^∞ if $\omega \neq 0$. They are in fact also regular at $\omega = 0$. For instance, using (34) and the fact that $a^0(t)$ is real, we can write

$$a^0(t) = \sum_\gamma (\cos \gamma\omega t - 1)u_\gamma{}^0 + 1$$

This form of $a^0(t)$ shows clearly that there is no singularity at $\omega = 0$. Hence, according to (56) and (57), the integrand of $F(X)$ is a C^∞ function of X. [Alternatively, one can deduce from the Dyson series of the classical propagator that (56) and (57) are entire functions of R and S^0.] Since the time dependence appears in phase factors, the lemma follows. ■

In order to estimate the difference between F_R^N and $F_R(Y_N)$, it is useful to introduce an approximate operator solution $\bar{\sigma}_{pN}(t)$ of the Heisenberg equation of motion (10). The solution $\bar{\sigma}_{pN}(t)$ is defined by the same formulas (31)–(33) as the semiclassical solution, where we replace S^α by S_N^α ($\alpha = +, -, 0$) and ω by

$$\omega_N = \mu[(S_N^0)^2 + 2(S_N^+S_N^- + S_N^-S_N^+)]^{1/2} \tag{C1}$$

In this definition, we make an arbitrary choice of the order of the noncommuting factors (notice, however, that ω_N, being proportional to the length of the total spin angular momentum, commutes with the S_N^α). We denote by $\bar{F}_R^N$ the operator that is analogous to F_R^N but calculated with the help of the approximate solution $\bar{\sigma}_{pN}(t)$.

Lemma 5. If the bath correlation functions $C_1(t)$ and $C_2(t)$ are $O(e^{-\gamma t})$ (with γ determined below), then

$$\bar{F}_R^N - F_R^N = O(1/N) \tag{C2}$$

Proof. It is easy to check that $\bar{\sigma}_{pN}(t)$ is a solution of (10) up to a term of order t/N, that is

$$\|(d/dt)\bar{\sigma}_{pN}(t) - i\Gamma_N(t)\bar{\sigma}_{pN}(t)\| \leqslant (Cst)t/N \tag{C3}$$

Indeed, writing down explicitly the left-hand side of (C3), one sees that it is identically zero after the permutation of a certain (finite) number of factors S_N^α ($\alpha = +, -, 0$) and $\exp(iS_N^0 t)$. All the involved commutators are $O(1/N)$ or $O(t/N)$ in operator norm.

The difference between $\sigma_{pN}(t)$ and the approximate solution satisfies the differential equation

$$\frac{d}{dt}[\sigma_{pN}(t) - \bar{\sigma}_{pN}(t)] = \left[i\Gamma_N(t)\bar{\sigma}_{pN}(t) - \frac{d}{dt}\bar{\sigma}_{pN}(t)\right] + i\Gamma_N(t)[\sigma_{pN}(t) - \bar{\sigma}_{pN}(t)]$$

Since $\|\Gamma_N(t)\| \leqslant \gamma_0$ is bounded uniformly with respect to N and t, this leads immediately to the estimate

$$\|\sigma_{pN}(t) - \bar{\sigma}_{pN}(t)\| \leqslant Cst(t^2/N)e^{\gamma_0 t} \tag{C4}$$

By the very definition of $\bar{F}_R^N$, (C4) implies (C2) if the hypothesis in the lemma holds true with some $\gamma > \gamma_0$. ■

In order to compare now $\bar{F}_R^N$ with the classical function $F_R(Y)$, we consider the operator defined in the functional calculus of the commuting observables R_N and S_N^0 obtained by replacing S^0 by S_N^0 and ω by ω_N, (C1), in $F_R(Y)$. We denote it by $F_R(Y_N)$.

One sees by inspection that the integrands of $\bar{F}_R^N$ and $F_R(Y_N)$ differ again by the order of a finite number of factors whose commutators are $O(1/N)$ or $O(t/N)$. Thus

$$\bar{F}_R^N - F_R(Y_N) = O(1/N) \tag{C5}$$

The conjunction of (C2) and (C5) gives the desired result (107).

Remark. F_R^N, $\bar{F}_R^N$, and $F_R(Y_N)$ can be considered as functions in the functional calculus of R_N and S_N^0. Expressions (107), (C2), and (C5) have to be understood as pointwise estimates of these functions.

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